SKLYANIN ALGEBRAS AND HILBERT SCHEMES OF POINTS

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Abstract. We construct projective moduli spaces for torsion-free sheaves on noncommutative projective planes. These moduli spaces vary smoothly in the parameters describing the noncommutative plane and have good properties analogous to those of moduli spaces of sheaves over the usual (commutative) projective plane $\mathbf{P}^2$.

The generic noncommutative plane corresponds to the Sklyanin algebra $S = \mathrm{Skl}(E,\sigma)$ constructed from an automorphism σ of infinite order on an elliptic curve $E \subset \mathbf{P}^2$. In this case, the fine moduli space of line bundles over S with first Chern class zero and Euler characteristic $1-n$ provides a symplectic variety that is a deformation of the Hilbert scheme of n points on $\mathbf{P}^2 \smallsetminus E$.

Contents

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1. Introduction

It has become apparent that vector bundles and instantons on noncommutative spaces play an important role in representation theory (for example, via quiver varieties [Na, BGK1, BGK2, Gi, LB2]), string theory [CDS, NSc, SW, KKO], and integrable systems [BW1, BW2]. In this paper we make the first systematic application of moduli-theoretic techniques from algebraic geometry to a problem in this area, the study of coherent sheaves on noncommutative projective planes.

The circle of ideas leading to this paper begins with the work of Cannings and Holland on *the first Weyl algebra* (or ring of differential operators on the affine line) $A_1 = \mathcal{D}(\mathbf{A}^1) = \mathbb{C}\{x, \partial\}/(\partial x - x\partial - 1)$. [CH] gives a (1-1) correspondence between isomorphism classes of right ideals of A_1 and points of a certain infinite dimensional Grassmannian. Curiously, this space appears in a completely different context: it is precisely the *adelic Grassmannian* $\mathrm{Gr}^{\mathrm{ad}}$ that plays an essential rôle in Wilson's study of rational solutions of the KP hierarchy of soliton theory [Wi]. The space $\mathrm{Gr}^{\mathrm{ad}}$ is related to a number of other interesting spaces. In particular it decomposes into a disjoint union $\mathrm{Gr}^{\mathrm{ad}} = \coprod_{n\geq 0} \mathcal{C}_n$ of certain quiver varieties $\mathcal{C}_n$, the *completed Calogero-Moser phase spaces*. The varieties $\mathcal{C}_n$ appear naturally in yet another way, as deformations of the Hilbert scheme $(\mathbf{A}^2)^{[n]}$ of n points in the affine plane (this deformation is the twistor family of the hyperkähler metric on $(\mathbf{A}^2)^{[n]}$).

This suggests that there might be a natural way to relate right ideals of A_1 more directly to both $\mathcal{C}_n$ and $(\mathbf{A}^2)^{[n]}$. This is done in [LB1, BW1, BGK1, BGK2], where noncommutative projective geometry is used to refine the work of [CH]. As will be explained shortly, there is a noncommutative projective plane $\mathbf{P}^2_\hbar$ and a "restriction" functor from the category of modules on $\mathbf{P}^2_\hbar$ to the category of A_1-modules. Under this map, isomorphism classes of right ideals of A_1 are in (1-1) correspondence with isomorphism classes of line bundles on $\mathbf{P}^2_\hbar$ that are "trivial at infinity." These line bundles have a natural integer invariant $c_2 \geq 0$ analogous to the second Chern class and the line bundles with $c_2 = n$ are then in bijection with points of $\mathcal{C}_n$. This corresponds nicely with the results of [CH] and [Wi]. Indeed, if one regards $\mathbf{P}^2_\hbar$ as a deformation of $\mathbf{P}^2$, then A_1 provides a deformation of the ring of functions on $\mathbf{A}^2 \subset \mathbf{P}^2$ and the line bundles on $\mathbf{P}^2_\hbar$ with $c_2 = n$ correspond naturally to points of a deformation of $(\mathbf{A}^2)^{[n]}$.

The classification of line bundles on $\mathbf{P}^2_\hbar$ also has an analogue for vector bundles of rank greater than one [KKO]. In this case, the classification is related to influential work of Nekrasov and Schwarz [NSc] in string theory concerning instantons on a noncommutative $\mathbb{R}^4$.

This story raises a number of problems:

(1) The plane $\mathbf{P}^2_\hbar$ is one of several families of noncommutative planes. Construct moduli spaces that classify vector bundles or even torsion-free coherent sheaves on all such planes. Show, in particular, that the classification results of [BW2, BGK1, KKO] come from moduli space structures.
(2) Prove that these moduli spaces behave well in families. For example, when the noncommutative plane is a deformation of (the category of coherent sheaves on) $\mathbf{P}^2$, this should provide deformations of the Hilbert schemes of points $(\mathbf{P}^2)^{[n]}$.
(3) The Hilbert scheme $(\mathbf{A}^2)^{[n]}$ has an algebraic symplectic structure induced by the hyperkähler metric. Construct Poisson or symplectic structures on

the analogous moduli spaces defined in (2) and study the resulting Poisson and symplectic geometry.

We solve all these problems in the present paper. For example, we obtain "elliptic" deformations of the Hilbert schemes of points $(\mathbf{P}^2)^{[n]}$ and $(\mathbf{P}^2 \smallsetminus E)^{[n]}$ for any plane elliptic curve E. The deformations $(\mathbf{P}^2)^{[n]}$ and $(\mathbf{P}^2 \smallsetminus E)^{[n]}$ do indeed carry Poisson and holomorphic symplectic structures respectively.

1.1. **Line bundles on Quantum Planes.** An introduction to the general theory of noncommutative projective geometry can be found in [St2, SV] and formal definitions are given in Section 2. Roughly speaking, and in accordance with Grothendieck's philosophy,[1] a noncommutative projective plane is an abelian category with the fundamental properties of the category $\mathrm{coh}(\mathbf{P}^2)$ of coherent sheaves on the projective plane. These categories have been classified and they have a surprisingly rich geometry.

To be more precise, let $S = k \oplus \bigoplus_{i \geq 1} S_i$ be a connected graded noetherian k-algebra over a field k. For simplicity we assume that $\mathrm{char}\, k = 0$ in the introduction, although many of our results are proved for general fields. By analogy with the commutative case one regards qgr-S, the category of finitely generated graded S-modules modulo those of finite length, as the category of coherent sheaves on the noncommutative (and nonexistent) projective variety $\mathrm{Proj}(S)$. As is explained in [SV, Section 11], the noncommutative analogues of $\mathrm{coh}(\mathbf{P}^2)$ are exactly the categories of the form qgr-S where S is an AS (Artin-Schelter) regular algebra with the Hilbert series $(1-t)^{-3}$ of a polynomial ring in three variables (to denote which we write $S \in \underline{\mathrm{AS}}_3$; see Definition 2.2). Accordingly, we will refer to qgr-S for such an algebra S as a *noncommutative* or *quantum projective plane*.

The algebras $S \in \underline{\mathrm{AS}}_3$ have in turn been classified in terms of geometric data [ATV1]. When qgr-$S \not\simeq \mathrm{coh}(\mathbf{P}^2)$, the classification is in terms of commutative data (E, σ), where $E \hookrightarrow \mathbf{P}^2$ is a (possibly singular) plane cubic curve and $\sigma \in \mathrm{Aut}(E)$ is a non-trivial automorphism. Moreover, S is then determined by this data and so we write $S = S(E, \sigma)$. A key fact is that $\mathrm{coh}(E) \simeq \mathrm{qgr}(S/gS) \subset$ qgr-S for an element $g \in S_3$ that is unique up to scalar multiplication and this inclusion has a left adjoint of "restriction to E." When qgr-$S \simeq \mathrm{coh}(\mathbf{P}^2)$, there is no canonical choice of E. In order to incorporate the categories qgr-$S \simeq \mathrm{coh}(\mathbf{P}^2)$ into the same framework, we identify such a category qgr-S with $\mathrm{coh}(\mathbf{P}^2)$, choose any cubic curve $E \subset \mathbf{P}^2$, and set $\sigma = \mathrm{Id}$; thus $S(E, \sigma) = k[x, y, z]$ in this case.

It follows from the classification that there is a rich supply of noncommutative planes (see Remark 2.9). In particular the plane $\mathbf{P}^2_\hbar$ associated to the Weyl algebra equals qgr-U, where $U = \mathbb{C}\{x, y, z\}/(yx - xy - z^2, z \text{ central})$ is an algebra $S(E, \sigma)$ for which $E = \{z^3 = 0\}$ is the triple line at infinity. Perhaps the most interesting and subtle algebra in $\underline{\mathrm{AS}}_3$ is the generic example: the *Sklyanin algebra* $\mathrm{Skl}(E, \sigma) = S(E, \sigma)$ which is determined by a smooth elliptic curve E and an automorphism σ given by translation under the group law. One can regard qgr-Skl as an elliptic deformation of $\mathrm{coh}(\mathbf{P}^2)$ and so we call qgr-Skl an *elliptic quantum plane*.

As has been mentioned, noncommutative projective planes have all the basic properties of $\mathrm{coh}(\mathbf{P}^2)$ and therefore admit natural definitions of vector bundles and torsion-free sheaves as well as invariants like Euler characteristics and Chern

[1] "In short, as A. Grothendieck taught us, to do geometry you really don't need a space, all you need is a category of sheaves on this would-be space" [Ma, p.83].

classes. In particular, *a vector bundle* in qgr-S is just the image of a reflexive graded S-module (see Definition 3.10 for a more homological definition in qgr-S). A *line bundle* is then a vector bundle of rank one. For any torsion-free module $\mathcal{L} \in$ qgr-S of rank one, a unique shift $\mathcal{L}(n)$ of $\mathcal{L}$ has *first Chern class* $c_1 = 0$ (see page 16). *The Euler characteristic* of $\mathcal{M} \in$ qgr-S is defined just as in the commutative case: $\chi(\mathcal{M}) = \sum(-1)^j \dim_k \mathrm{H}^j(\mathcal{M})$, where $\mathrm{H}^j(\mathcal{M}) = \mathrm{Ext}^j_{\text{qgr-}S}(S, \mathcal{M})$.

We can now describe our main results on the structure of rank 1 modules in qgr-S. Their significance will be described later in the introduction.

Theorem 1.1 (Theorem 8.11)**.** *Let* qgr-S*, for $S = S(E,\sigma)$, be a noncommutative projective plane. Then*

(1) *There is a smooth, projective, fine moduli space $\mathcal{M}^{ss}_S(1,0,1-n)$ for rank one torsion-free modules in* qgr-S *with $c_1 = 0$ and $\chi = 1-n$. Moreover,* $\dim \mathcal{M}^{ss}_S(1,0,1-n) = 2n$.

(2) $\mathcal{M}^{ss}_S(1,0,1-n)$ *has a nonempty open subspace $(\mathbf{P}_S \smallsetminus E)^{[n]}$ parametrizing modules whose restriction to E is torsion-free. When $|\sigma| = \infty$, $(\mathbf{P}^2 \smallsetminus E)^{[n]}$ parametrizes line bundles with $c_1 = 0$ and $\chi = 1-n$.*

Remark 1.2. (1) When $\mathcal{M} \in$ qgr-S is torsion-free of rank one and $c_1(\mathcal{M}) = 0$, the formula for $\chi(\mathcal{M})$ simplifies to $\chi(\mathcal{M}) = 1 - \dim_k \mathrm{H}^1(\mathcal{M}(-1))$ (see Corollary 6.6).

(2) We emphasize that the moduli spaces in this theorem (and all other results of this paper) are schemes in the usual sense; they are constructed as GIT quotients of subvarieties of a product of Grassmannians—see Theorem 7.7.

Most of the algebras in $\underline{\mathrm{AS}}_3$ (including the homogenized Weyl algebra U, the Sklyanin algebra Skl and the various quantum spaces defined by three generators x_i which q-commute $x_i x_j = q_{ij} x_j x_i$) occur in families that include the commutative polynomial ring. For these algebras, the moduli spaces behave well in families (see Theorem 1.7, below) and we can improve Theorem 1.1 as follows:

Theorem 1.3 (Theorem 8.12)**.** *Let $\mathcal{B}$ be a smooth irreducible curve defined over k and let $S_{\mathcal{B}} = S_{\mathcal{B}}(E,\sigma) \in \underline{\mathrm{AS}}_3$ be a flat family of algebras such that $S_p = k[x,y,z]$ for some point $p \in \mathcal{B}$. Let $S = S_b$ for any $b \in \mathcal{B}$. Then*

(1) $\mathcal{M}^{ss}_S(1,0,1-n)$ *and $(\mathbf{P}_S \smallsetminus E)^{[n]}$ are irreducible and hence connected.*

(2) $\mathcal{M}^{ss}_S(1,0,1-n)$ *is a deformation of the Hilbert scheme $(\mathbf{P}^2)^{[n]}$, with its subvariety $(\mathbf{P}_S \smallsetminus E)^{[n]}$ being a deformation of $(\mathbf{P}^2 \smallsetminus E)^{[n]}$.*

For the elliptic quantum planes, we get in addition:

Theorem 1.4 (Theorem 9.4)**.** *Assume that $S = \mathrm{Skl}(E,\sigma)$ is a Sklyanin algebra defined over $\mathbb{C}$. Then $\mathcal{M}^{ss}_S(1,0,1-n)$ admits a Poisson structure which restricts to a holomorphic symplectic structure on $(\mathbf{P}_S \smallsetminus E)^{[n]}$.*

When $S = \mathrm{Skl}(E,\sigma)$ for an automorphism σ of infinite order, de Naeghel and Van den Bergh [DV] have independently used quiver varieties to obtain a geometric description of the set of isomorphism classes of line bundles in qgr-S. One advantage of their approach is that it shows that, in stark contrast to the commutative case, $(\mathbf{P}_S \smallsetminus E)^{[n]}$ is affine.

Just as one can restrict modules from qgr-S to E, so there is a natural functor of "restriction to $\mathrm{Proj}(S) \smallsetminus E$." Formally, this means inverting the canonical element $g \in S(E,\sigma)_3$ defining E to obtain the localized algebra $A(S) = S[g^{-1}]_0$ that can be regarded as a noncommutative deformation of the ring of functions on $\mathbf{P}^2 \smallsetminus E$. The

restriction map from qgr-S to $A(S)$-mod takes a module M to $M[g^{-1}]_0$. When S is the algebra U, $A(U)$ is just the Weyl algebra A_1 and so the next result gives a natural analogue of the results of [CH, BW1, BW2, BGK1] discussed at the beginning of the introduction.

Theorem 1.5 (See Theorem 3.8). *Let $S = \mathrm{Skl}(E, \sigma)$. Then the closed points of*

$$\coprod_{s \in \mathbf{Z}/3\mathbf{Z}} \coprod_{n \geq 0} (\mathbf{P}_S \smallsetminus E)^{[n]} \tag{1.1}$$

are in bijective correspondence with the isomorphism classes of finitely generated rank 1 torsion-free $A(S)$-modules via the map $M \mapsto M[g^{-1}]_0$.

The appearance of $\mathbf{Z}/3\mathbf{Z}$ in Equation 1.1 corresponds to the appearance of $\mathrm{Pic}(\mathbf{P}^2 \smallsetminus E) = \mathbf{Z}/3\mathbf{Z}$ in the commutative classification. The bijection of Theorem 1.5 is *not* an isomorphism of moduli, nor should one hope for it to be one—even in the commutative case the moduli of sheaves on affine varieties are not well behaved. One should note that the analogue of Theorem 1.5 does not hold for all noncommutative planes: Proposition 3.15 gives a counterexample for a ring of q-difference operators.

While Theorem 1.5 holds for all values of the automorphism σ, there are really two distinct cases to the theorem. When $|\sigma| = \infty$, $A(S)$ is a simple hereditary ring and so the theorem classifies projective rank one modules, just as the work of Cannings-Holland et al. classified projective A_1-modules. However, when $|\sigma| < \infty$, $A(S)$ is an Azumaya algebra of dimension two and so the classification includes rank 1 torsion-free modules that are not projective.

1.2. **General Results and Higher Rank.** The moduli results from Theorems 1.1, 1.3 and 1.4 also have analogues for modules of higher rank that mimic the classical results for vector bundles and torsion-free sheaves on $\mathbf{P}^2$. As in the commutative case, one has a natural notion of (semi)stable modules (see page 30) and we prove results that are direct analogues of the commutative results as described, for example, in [DL, LP1] and [OSS].

Theorem 1.6 (Theorem 7.10). *Let* qgr-S *be a noncommutative projective plane and fix $r \geq 1$, $c_1 \in \mathbf{Z}$, and $\chi \in \mathbf{Z}$.*

1. *There is a projective coarse moduli space $\mathcal{M}_S^{ss}(r, c_1, \chi)$ for geometrically semistable torsion-free modules in* qgr-S *of rank r, first Chern class c_1 and Euler characteristic χ.*
2. *$\mathcal{M}_S^{ss}(r, c_1, \chi)$ has as an open subvariety the moduli space $\mathcal{M}_S^{s}(r, c_1, \chi)$ for geometrically stable modules in* qgr-S.

As the next result shows, the moduli spaces $\mathcal{M}_S^{s}(r, c_1, \chi)$ also behave well in families.

Theorem 1.7 (Theorem 8.1). *Suppose that $S = S_{\mathcal{B}}$ is a flat family of algebras in $\underline{\mathrm{AS}}_3$ parametrized by a k-scheme $\mathcal{B}$. Then there is a quasi-projective $\mathcal{B}$-scheme $\mathcal{M}_S^{s}(r, c_1, \chi) \to \mathcal{B}$ that is smooth over $\mathcal{B}$, and the fibre of which over $b \in \mathcal{B}$ is precisely $\mathcal{M}_{S_b}^{s}(r, c_1, \chi)$. In particular, $\mathcal{M}_S^{s}(r, c_1, \chi)$ is smooth when $S \in \underline{\mathrm{AS}}_3(k)$.*

In the case of the Weyl algebra, our methods give an easy proof that the parametrizations of Berest-Wilson [BW1] and Kapustin-Kuznetsov-Orlov [KKO] are really fine moduli spaces. The paper [KKO] constructs a variety $V /\!\!/ \mathrm{GL}(n)$

together with a bijection between the set of points of $V/\!\!/\mathrm{GL}(n)$ and the set of isomorphism classes of framed vector bundles in $\mathbf{P}^2_\hbar$ of rank r and Euler characteristic $1-n$ (this generalizes the bijection constructed by [BW1] for rank 1).

Proposition 1.8 (Proposition 8.13). *The variety $V/\!\!/\mathrm{GL}(n)$ is a fine moduli space for framed vector bundles of rank r and Euler characteristic $\chi = 1-n$ in $\mathbf{P}^2_\hbar$. This isomorphism induces the bijections of* [BW1, KKO].

1.3. **Deformations and Poisson Structures.** As we have seen, many of the noncommutative projective planes can be regarded as deformations of $\mathrm{coh}(\mathbf{P}^2)$ and so, by Theorem 1.3, they determine deformations of the Hilbert scheme of points on $\mathbf{P}^2$. We expand on this observation in this subsection. For simplicity we just the discuss the case of a Sklyanin algebra $S = \mathrm{Skl}(E,\sigma)$ defined over $\mathbb{C}$. We first note that Theorem 1.4 also generalizes to higher ranks:

Theorem 1.9 (Theorem 9.4). *Let $S = \mathrm{Skl}(E,\sigma)(\mathbb{C})$.*

1. *The moduli space $\mathcal{M}^s_S(r, c_1, \chi)$ admits a Poisson structure.*
2. *Fix a vector bundle H on E and let $\mathcal{M}_H$ denote the smooth locus of the locally closed subscheme of $\mathcal{M}^s_S(r, c_1, \chi)$ parametrizing modules $\mathcal{E}$ that satisfy $\mathcal{E}|_E \cong H$. Then the Poisson structure of $\mathcal{M}^s_S(r, c_1, \chi)$ restricts to a symplectic structure on $\mathcal{M}_H$.*

The elliptic curve E is the zero locus of a section s of $\mathcal{O}(3) = \Lambda^2 T_{\mathbf{P}^2}$ that is unique up to scalar multiplication; upon restriction to $\mathbf{P}^2 \smallsetminus E$, s^{-1} is an algebraic symplectic structure. The Poisson structure s also induces a Poisson structure on the Hilbert scheme $(\mathbf{P}^2)^{[n]}$ that restricts to give a symplectic structure on $(\mathbf{P}^2 \smallsetminus E)^{[n]}$ (see [Be]). This Poisson structure on $\mathbf{P}^2$ determines a deformation of the polynomial ring $k[x,y,z]$ that is precisely the Sklyanin algebra $\mathrm{Skl}(E,\sigma)$, with σ corresponding to the deformation parameter [Od, Appendix D2]. It is natural to hope that the noncommutative deformations of $\mathbf{P}^2$ induce deformations of $(\mathbf{P}^2)^{[n]}$ and $(\mathbf{P}^2 \smallsetminus E)^{[n]}$ that also carry Poisson and symplectic structures. The point of Theorems 1.3(2) and 1.4 is that $(\mathbf{P}^2_S)^{[n]}$ and $(\mathbf{P}^2_S \smallsetminus E)^{[n]}$ provide just such a deformation.

The relationship between the deformed Hilbert scheme of points $(\mathbf{P}^2_S)^{[n]}$ and collections of points in $\mathbf{P}^2$ or qgr-S is rather subtle (as was also the case with the original work on modules over the Weyl algebra). For simplicity, we will explain this when $|\sigma| = \infty$ and we regard $(\mathbf{P}^2_S)^{[n]}$ as parametrizing ideal sheaves $\mathcal{I} \subset \mathcal{O}_{\mathbf{P}^2}$ for which $\mathcal{O}_{\mathbf{P}^2}/\mathcal{I}$ has length n. Now the only simple objects in qgr-S are the modules corresponding to points on E.[2] By mimicking the commutative procedure, it is therefore easy to find the modules in $(\mathbf{P}^2_S)^{[n]}$ corresponding to collections of points on E, but the modules parametrized by $(\mathbf{P}^2_S \smallsetminus E)^{[n]}$ are necessarily more subtle (compare Corollary 7.17 with Proposition 8.9). In essence, the proof of Theorem 1.1(2) shows that the ideal sheaves parametrized by $(\mathbf{P}^2 \smallsetminus E)^{[n]}$ can be constructed as the cohomology of certain complexes (monads) that deform to complexes in qgr-S. The cohomology of such a deformed complex cannot correspond to a collection of points in qgr-S simply because these points do not exist. It is therefore forced to be an interesting line bundle. This subtlety is also reflected in Theorem 1.5 since one is parametrizing projective $A(S)$-modules rather than annihilators of finite dimensional modules.

[2] From the point of view of quantization, this corresponds to the fact that the Poisson structure on $\mathbf{P}^2$ is nondegenerate over $\mathbf{P}^2 \smallsetminus E$.

The Poisson structure of $\mathcal{M}^s_S(r, c_1, \chi)$ in Theorem 1.9 is related to work of Feigin-Odesskii [FO] on their generalizations $Q_{n,k}(E, \sigma)$ of higher dimensional Sklyanin algebras. The classical limit of the $Q_{n,k}(E, \sigma)$ induces a Poisson structure on the moduli spaces of certain vector bundles over E [FO, Theorem 1]. This structure is in turn a special case of a Poisson pairing on the moduli space of P-bundles on E, where P is a parabolic subgroup of a reductive group. Now, our moduli spaces may also be identified with moduli spaces of filtered E-vector bundles and the method we use to obtain our Poisson structure on $\mathcal{M}^s_s(r, c_1, \chi)$ is to use Polishchuk's generalization [Pl] of the Feigin-Odesskii construction (see Subsection 9.1). Given this connection with [FO] and the $Q_{n,k}(E, \sigma)$, it would be interesting to know whether our Poisson structure induces similarly interesting noncommutative deformations of $\mathcal{M}^s_s(r, c_1, \chi)$.

Finally, it would be interesting to relate our moduli spaces to integrable systems. We make a small step in this direction in Subsection 9.2 by explaining how the symplectic leaves of $\mathcal{M}^s_S(r, c_1, \chi)$ may be related to parameter spaces for Higgs bundles with values in the centrally extended current group of [EF].

1.4. **Methods.** One key element of the paper is the thoroughgoing use of generalizations of the cohomological tools from commutative algebraic geometry, primarily Cohomology and Base Change. These may be found in a form sufficient for our purposes in Section 4. In particular, it is through these methods that we are able to prove formal moduli results.

Our treatment of semistability and linearization of the group actions in Sections 6 and 7 follows the outline of Drezet and Le Potier [DL, LP1] for $\mathbf{P}^2$. Their techniques require some modification, however, because of the shortage of "points" on our noncommutative surfaces. This proof has several steps. First, we construct a version of the Beilinson spectral sequence. The version used here is different than that appearing in [LB1, BW1, KKO, BGK1], since it uses the Čech complex from [VW1] to avoid the problem that tensor products of modules over a noncommutative ring are no longer modules. This spectral sequence is then used to show that the moduli space of semistable modules in qgr-S is equivalent to that for semistable monads and then for semistable Kronecker complexes. Here, *a Kronecker complex* is a complex $S(-1)^a \xrightarrow{\alpha} S^b \xrightarrow{\beta} S(1)^c$ in qgr-S. This complex is *a monad* if α is injective and β is surjective. The semistable Kronecker complexes can be described by purely commutative data and are easy to analyse by standard techniques of GIT quotients. An important fact is that this all works in families, which is why we are able to construct our moduli spaces.

The construction of a Poisson structure and the relation with Higgs bundles are analogs of results of [Pl] and [GM], respectively.

Since we hope that the paper may be of interest to readers of varied backgrounds, we have included some details in proofs that we imagine will be useful for some readers but unnecessary for others.

1.5. **Further Directions.** A natural question that is not answered in the present work concerns the metric geometry of our deformations of Hilbert schemes. The plane cubic complement $\mathbf{P}^2 \smallsetminus E$ admits a complete hyperkähler metric (we are grateful to Tony Pantev and Kevin Corlette for explaining this to us) and one imagines that the Hilbert schemes of points also admit such metrics; if this is true,

one would like to know the relationship between our deformations and the twistor family for the hyperkähler metric on the Hilbert scheme.

In a more speculative direction, one would like to have an interpretation, parallel to that in [BW1, BW2, BGK1, BGK2], of the moduli spaces $\mathcal{M}_S^{ss}(r, c_1, \chi)$ in terms of integrable systems. Section 9 makes a start in this direction, but this is certainly an important direction for further work to which we hope to return.

As was remarked earlier, the Weyl algebra and its homogenization have been applied to questions in string theory [KKO, NSc] and so it would be interesting to understand whether the other quantum planes have applications in this direction [DN, Section VI.B]. The Sklyanin algebra appears in general marginal deformations of $N = 4$ super Yang-Mills theory (see [BJL, Equations 2.5–2.7 and Section 4.6.1]) and, in the terminology of that paper, Theorems 1.1 and 1.3 can be interpreted as a parametrization of space-filling D-branes (see [BL, Section 6]).

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2. Background Material

In this section we collect the basic definitions and results from the literature that will be used throughout the paper.

Fix a base field k and a commutative k-algebra C. Noncommutative projective geometry [St2, SV] is concerned with the study of connected graded (cg) k-algebras or, more generally cg C-algebras, where a graded C-algebra $S = \bigoplus_{i \geq 0} S_i$ is called *connected graded* if $S_0 = C$ is a central subalgebra. The nth *Veronese ring* is defined to be the ring $S^{(n)} = \bigoplus_{i \geq 0} S_{ni}$, graded by $S_i^{(n)} = S_{ni}$. Write Mod-S for the category of right S-modules and Gr-S for the category of graded right S-modules, with homomorphisms $\mathrm{Hom}(M, N) = \mathrm{Hom}_S(M, N)$ being graded homomorphisms of degree zero. Given $M = \bigoplus_{i \in \mathbb{Z}} M_i$, the *shift* $M(n)$ of M is the graded module $M(n) = \bigoplus M(n)_i$ defined by $M(n)_i = M_{i+n}$ for all i. The other hom group is $\underline{\mathrm{Hom}}_S(M, N) = \bigoplus_{r \in \mathbb{Z}} \mathrm{Hom}(M, N(r))$, with derived functors $\underline{\mathrm{Ext}}^j(M, N)$. If M is finitely generated, then $\underline{\mathrm{Ext}}^j(M, N) = \mathrm{Ext}^j_{\text{Mod-}S}(M, N)$. Similar conventions apply to Tor groups.

A module $M = \bigoplus_{i \in \mathbb{Z}} M_i \in$ Gr-S is called *right bounded* if $M_i = 0$ for $i \gg 0$. The full Serre subcategory of Gr-S generated by the right bounded modules is denoted Tors-S with quotient category Qgr-S = Gr-S/ Tors-S. If S happens to be noetherian (which will almost always be the case in this paper) write mod-S, gr-S and qgr-S = gr-S/ tors-S for the subcategories of noetherian objects in these three categories. Observe that tors-S is just the category of finite-dimensional graded modules. Similar definitions apply for left modules and we write the corresponding categories as S-gr, etc.

We now turn to the definitions for the algebras of interest to us: Sklyanin algebras and, more generally, Artin-Schelter regular rings. Fix a (smooth projective) elliptic curve $\iota : E \hookrightarrow \mathbf{P}^2$ with corresponding line bundle $\mathcal{L} = \iota^*(\mathcal{O}_{\mathbf{P}^2}(1))$ of degree 3. Fix an automorphism $\sigma \in \mathrm{Aut}(E)$ given by translation under the group law and denote the graph of σ by $\Gamma_\sigma \subset E \times E$. If $V = \mathrm{H}^0(E, \mathcal{L})$, there is a 3-dimensional space

$$\mathcal{R} = \mathrm{H}^0\left(E \times E, (\mathcal{L} \boxtimes \mathcal{L})(-\Gamma_\sigma)\right) \subset \mathrm{H}^0\left(E \times E, \mathcal{L} \boxtimes \mathcal{L}\right) = V \otimes V. \tag{2.1}$$

Definition 2.1. The 3-*dimensional Sklyanin algebra* is the algebra

$$S = \mathrm{Skl} = \mathrm{Skl}(E, \mathcal{L}, \sigma) = T(V)/(\mathcal{R}),$$

where $T(V)$ denotes the tensor algebra on V.

When σ is the identity, $\mathcal{R} = \Lambda^2 V$ and so $\mathrm{Skl}(E, \mathcal{L}, \mathrm{Id})$ is just the polynomial ring $k[x, y, z]$. One can therefore regard $\mathrm{Skl}(E, \mathcal{L}, \sigma)$ as a deformation of that polynomial ring and this deformation is flat [TV]. One may also write $\mathrm{Skl}(E, \mathcal{L}, \sigma)$ as the k-algebra with generators x_1, x_2, x_3 and relations:

$$ax_ix_{i+1} + bx_{i+1}x_i + cx_{i+2}^2 = 0 \qquad i = 1, 2, 3 \mod 3, \tag{2.2}$$

where $a, b, c \in k^*$ are any scalars satisfying $(3abc)^3 \neq (a^3 + b^3 + c^3)^3$ (see the introduction to [ATV1]).

Basic properties of $S = \mathrm{Skl}(E, \mathcal{L}, \sigma)$ can be found in [ATV1, ATV2] and are summarized in [SV, Section 8]. In particular, it is one of the most important examples of an AS regular ring defined as follows:

Definition 2.2. An *Artin-Schelter (AS) regular ring of dimension three* is a connected graded algebra S satisfying

- S has global homological dimension 3, written $\mathrm{gldim}\, S = 3$;
- S has Gelfand-Kirillov dimension 3, written $\mathrm{GKdim}\, S = 3$; and
- S satisfies the *AS Gorenstein condition*: $\underline{\mathrm{Ext}}^i(k, S) = \delta_{i,3}k(\ell)$, for some ℓ.

Let $\underline{\mathrm{AS}}_3 = \underline{\mathrm{AS}}_3(k)$ denote the class of AS regular k-algebras of dimension three with Hilbert series $(1 - t)^{-3}$ and with $\ell = -3$.

The algebras in $\underline{\mathrm{AS}}_3$ have been classified in [ATV1] and we will list the generic examples at the end of this section (see Remark 2.9). In particular, $\mathrm{Skl} \in \underline{\mathrm{AS}}_3$. As is justified in [SV, Section 11], the category qgr-S for $S \in \underline{\mathrm{AS}}_3$ can be regarded as a (nontrivial) noncommutative projective plane and most of our results work for qgr-S for any $S \in \underline{\mathrm{AS}}_3$.

There are two basic techniques for understanding S and Qgr-S for $S \in \underline{\mathrm{AS}}_3$. First, assume that $S \in \underline{\mathrm{AS}}_3$ but that Qgr-S is *not* equivalent to $\mathrm{Qcoh}(\mathbf{P}^2)$, the category of quasi-coherent sheaves on $\mathbf{P}^2$. By [ATV1, Theorem II] there exists, up to a scalar multiple, a canonical normal element $g \in S_3$ (thus, $gS = Sg$). The factor ring S/gS is isomorphic to the twisted homogeneous coordinate ring $B = B(E, \mathcal{L}, \sigma)$ for a cubic curve $\iota : E \hookrightarrow \mathbf{P}^2$, the line bundle $\mathcal{L} = \iota^*\mathcal{O}_{\mathbf{P}^2}(1)$ of degree three and an automorphism σ of the scheme E. The details of this construction can be found in [AV] or [SV, Section 3], but the essential properties are the following: write $\mathcal{N}^\tau$ for the pull-back of a sheaf $\mathcal{N}$ along an automorphism τ of E and set $\mathcal{L}_n = \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}}$. Then $B = \bigoplus_{n \geq 0} B_n$, where $B_n = \mathrm{H}^0(E, \mathcal{L}_n)$. The algebra S is determined by the data $(E, \mathcal{L}, \sigma)$ and so we write $S = S(E, \mathcal{L}, \sigma)$. When $S = \mathrm{Skl}$, E and σ coincide with the objects used in Definition 2.1. In fact, $\mathcal{L}$ is superfluous to the classification, both for the Sklyanin algebra and in general. However, this sheaf appears in many of our results and so we have included it in the notation.

We are mostly interested in Qgr-S rather than S itself, and it is useful to be able to use the curve E. However there is no canonical embedded curve when Qgr-$S \simeq \mathrm{Qcoh}(\mathbf{P}^2)$. Thus, in order to have a uniform approach, we let $\underline{\mathrm{AS}}'_3$ denote the algebras $S \in \underline{\mathrm{AS}}_3$ for which Qgr-S is *not* equivalent to $\mathrm{Qcoh}(\mathbf{P}^2)$, together with $S = k[x, y, z]$. For $S = k[x, y, z]$ we fix any nonzero homogeneous element $g \in S_3$,

set $E = \mathrm{Proj}(S/gS)$ and, somewhat arbitrarily, write $S = S(E, \mathcal{L}, \mathrm{Id})$, where $\mathcal{L}$ is the restriction of $\mathcal{O}_{\mathbf{P}^2}(1)$ to E. Thus, for each $S \in \underline{\mathrm{AS}}_3$ there exists $S' \in \underline{\mathrm{AS}}'_3$ with $\mathrm{Qgr}\text{-}S \simeq \mathrm{Qgr}\text{-}S'$.

Let $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$. The factor ring $B = B(E, \mathcal{L}, \sigma)$ is noetherian. More significantly, $\mathrm{Qcoh}(E) \simeq \mathrm{Qgr}\text{-}B$ via the map

$$\xi : \mathcal{F} \mapsto \bigoplus_{i \geq 0} \mathrm{H}^0(E, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}_n) \qquad \text{for} \quad \mathcal{F} \in \mathrm{Qcoh}(E).$$

This induces a map $\rho : \mathrm{Qgr}\text{-}S \to \mathrm{Qgr}\text{-}S/gS = \mathrm{Qgr}\text{-}B \simeq \mathrm{Qcoh}(E)$ and, mimicking geometric notation, we write

$$\mathcal{M}|_E = \rho(\mathcal{M}) \in \mathrm{Qcoh}(E) \qquad \text{for} \quad \mathcal{M} \in \mathrm{Qgr}\text{-}S. \tag{2.3}$$

When $S = \mathrm{Skl}$, the element g is central and B is a domain. For other $S \in \underline{\mathrm{AS}}_3$, it can happen that g is not central or that E is not integral. In the latter case B will not be a domain.

The second important technique in the study of qgr-S, for $S \in \underline{\mathrm{AS}}_3$, is to use cohomological techniques modelled on the classical case. This works well is because S satisfies the χ condition of [AZ1, Definition 3.7], defined as follows: a cg C-algebra R satisfies χ if $\mathrm{Ext}^j_{\mathrm{Mod}\text{-}R}(R/R_{\geq n}, M)$ is a finitely generated C-module for all $M \in \mathrm{Mod}\text{-}R$, all $j \geq 0$, and all $n \gg 0$. Write π for the natural projections $\mathrm{Gr}\text{-}S \to \mathrm{Qgr}\text{-}S$ and $\mathrm{gr}\text{-}S \to \mathrm{qgr}\text{-}S$ and set $\pi(S) = \mathcal{O}_S$. One may pass back from Qgr-S to Gr-S by the "global sections" functor:

$$\Gamma^*(\mathcal{M}) = \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathrm{Qgr}\text{-}S}(\mathcal{O}, \mathcal{M}(n)), \qquad \text{for } \mathcal{M} \in \mathrm{Qgr}\text{-}S. \tag{2.4}$$

One then has an adjoint pair (π, Γ^*). We will occasionally use the module $\Gamma(\mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Qgr}\text{-}S}(\mathcal{O}, \mathcal{M}(n))$. This module has the disadvantage that $\Gamma(\pi(M))$ need not be finitely generated, but $\Gamma(\mathcal{M})/\Gamma^*(\mathcal{M})$ is at least right bounded.

As in the commutative situation, we will write the derived functors of $\mathrm{H}^0(\mathcal{M}) = \mathrm{Hom}_{\mathrm{Qgr}\text{-}S}(\mathcal{O}, \mathcal{M})$, for $\mathcal{M} \in \mathrm{Qgr}\text{-}S$, as

$$\mathrm{H}^i(\mathcal{M}) = \mathrm{H}^i(\mathrm{Qgr}\text{-}S, \mathcal{M}) = \mathrm{Ext}^i_{\mathrm{Qgr}\text{-}S}(\mathcal{O}, \mathcal{M}).$$

One also has the analogous objects for qgr-B and one should note that there is no confusion in the notation since $\mathrm{H}^i(\mathrm{Qgr}\text{-}S, \mathcal{M}) = \mathrm{H}^i(E, \mathcal{M})$ for $i \geq 0$ and $\mathcal{M} \in \mathrm{qgr}\text{-}B$ (see [AZ1, Theorem 8.3]).

The significance of the χ condition is that the functors H^i are particularly well-behaved. In particular, [AZ1, Theorems 8.1 and 4.5 and Corollary, p.253] combine to show:

Lemma 2.3. *Let $S \in \underline{\mathrm{AS}}_3$. Then:*

1. $\pi\Gamma^*(\mathcal{M}) = \mathcal{M}$ *for any* $\mathcal{M} \in \mathrm{Qgr}\text{-}S$.
2. *If* $M \in \mathrm{gr}\text{-}S$, *then* $\Gamma^*(\pi M) = M$ *up to a finite dimensional vector space.*
3. *More precisely, if* $M = \bigoplus_{i \geq 0} M_i \in \mathrm{gr}\text{-}S$ *and* F *is the maximal finite-dimensional submodule of* M, *then* $\Gamma^*\pi M$ *is the maximal positively graded essential extension of* M/F *by a finite-dimensional module.*
4. $\Gamma^*(\pi(S)) = S = \Gamma(\pi(S))$ *and* $\Gamma^*(\pi(S/gS)) = S/gS = \Gamma(\pi(S/gS))$ *whenever* $S \in \underline{\mathrm{AS}}'_3$.
5. $\mathrm{H}^1(\mathcal{O}_S(j)) = 0$ *for all* $j \in \mathbb{Z}$ *and* $\mathrm{H}^2(\mathcal{O}_S(k)) = 0$ *for all* $k \geq -2$.
6. S *has* cohomological dimension 2 *in the sense that* $\mathrm{H}^i(\mathcal{M}) = 0$ *for all* $\mathcal{M} \in \mathrm{Qgr}\text{-}S$ *and* $i > 2$. □

There is also an analogue of Serre duality:

Proposition 2.4 (Serre Duality)**.** *Let $S \in \underline{\mathrm{AS}}_3$ and $\mathcal{M}, \mathcal{N} \in$ qgr-S. Then*
$$\mathrm{Ext}^i(\mathcal{M}, \mathcal{N}) \cong \mathrm{Ext}^{2-i}(\mathcal{N}, \mathcal{M}(-3))^*.$$

Proof. By [YZ, Theorem 2.3] and [AZ1, Theorem 8.1(3)] one has
$$\mathrm{Ext}^i(\mathcal{M}, \mathcal{O}(-3)) \cong \mathrm{H}^{2-i}(\mathcal{M})^* = \mathrm{Ext}^{2-i}(\mathcal{O}, \mathcal{M})^*$$
for all $\mathcal{M} \in$ qgr-S. This version of Serre duality implies that $\mathrm{Ext}^2(\mathcal{M}, \mathcal{O}(-n)) \neq 0$ for $n \gg 0$, and $\mathrm{Ext}^i(\mathcal{M}, \mathcal{O}(-n)) = 0$ for $n \gg 0$ and $i < 2$. Also, since S has finite global homological dimension, Lemma 2.3(6) implies that $\mathrm{Ext}^j(\mathcal{M}, \mathcal{Q}) = 0$ for all $\mathcal{Q} \in$ Qgr-S and $i > 2$. Set $\mathcal{L} = \pi \underline{\mathrm{Ext}}^2(\mathcal{M}, \mathcal{O}(-3))^* \cong \mathcal{M}(-3)$. Then all the hypotheses of [YZ, Theorem 2.2] are satisfied and that result implies that $\mathrm{Ext}^i(\mathcal{M}, \mathcal{N}) \cong \mathrm{Ext}^2(\mathcal{N}, \mathcal{L})^* \cong \mathrm{Ext}^{2-i}(\mathcal{N}, \mathcal{M}(-3))^*$. □

The next two lemmas give some standard facts about the relationship between modules over S and E for which we could not find a convenient reference.

Lemma 2.5. *Let $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$. Suppose that $\mathcal{M}$ is a module in* qgr-S *and that $j \in \mathbb{Z}$. Then $\mathcal{M}(j)|_E = (\mathcal{M}|_E)^{\sigma^{-j}} \otimes_{\mathcal{O}_E} \mathcal{L}_j^{\tau_j}$, where $\tau_j = \sigma^{-j}$ if $j \geq 0$ but $\mathcal{L}_j = \mathcal{L}_{-j}^{-1}$ and $\tau_j = 1$ if $j < 0$.*

Proof. Let $\mathcal{F} \in \mathrm{coh}(E)$, with $\xi(\mathcal{F}) = \bigoplus_{j \geq 0} \mathrm{H}^0(\mathcal{F} \otimes \mathcal{L}_j) \in$ gr-B. By [SV, (3.1)] the shift functor in gr-B can be reinterpreted in terms of $\mathcal{O}_E$-modules as:
$$\xi(\mathcal{F})(n) = \xi\left(\sigma_*^n(\mathcal{F} \otimes \mathcal{L}_n)\right) = \xi\left(\mathcal{F}^{\sigma^{-n}} \otimes \mathcal{L}_n^{\sigma^{-n}}\right) \qquad \text{for } n \geq 0.$$
A simple computation then shows that $\xi(\mathcal{F})(-n) = \xi\left(\mathcal{F}^{\sigma^n} \otimes \mathcal{L}_n^{-1}\right)$ for $n \geq 0$. Since restriction to E commutes with the shift functor, the lemma follows. □

Let $S \in \underline{\mathrm{AS}}_3$. A module $M \in$ Gr-S is called *torsion-free*, respectively *torsion*, if no element, respectively every element, of M is killed by a nonzero element of S. A module $\mathcal{M} \in$ Qgr-S is *torsion-free* if $\mathcal{M} = \pi(M)$ for some torsion-free module $M \in \mathrm{Gr}\, S$. By [AZ1, S2, p.252] this is equivalent to $\Gamma^*(\mathcal{M})$ being torsion-free. The term "torsion" is also used in the literature to mean elements of Tors-S, but that will never be the case in this paper. A torsion-free module $\mathcal{M}$ in (qgr-S or gr-S) has *rank* m if $\mathcal{M}$ has Goldie rank m; that is, $\mathcal{M}$ contains a direct sum of m, but not $m+1$, nonzero submodules. Note that $\mathcal{M} \in$ qgr-S is torsion-free of rank one if and only if $\Gamma^*(\mathcal{M})$ is isomorphic to a shift of a right ideal of S.

Lemma 2.6. *Let $S \in \underline{\mathrm{AS}}'_3$ and suppose that $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence in* qgr-S *such that $\mathcal{F}_3$ is torsion-free. Then the restriction of this sequence to E remains exact.*

Proof. Since Γ^* is left exact, we have an exact sequence of S-modules
$$0 \to F_1 \to F_2 \to F_3 \to 0, \tag{2.5}$$
where $F_i = \Gamma^*(\mathcal{F}_i)$ for $i = 1, 2$ and F_3 is the image of $\Gamma^*(\mathcal{F}_2)$ in $\Gamma^*(\mathcal{F}_3)$. By [AZ1, S2, p.252] F_3 is torsion-free and, by [AZ1, Corollary, p.253], $\Gamma^*(\mathcal{F}_3)/F_3$ is finite dimensional. As F_3 is torsion-free, $F_3 \otimes gS \hookrightarrow F_3 \otimes S$ and so $\underline{\mathrm{Tor}}^1(F_3, S/gS) = 0$. Therefore, (2.5) induces the exact sequence:
$$0 \to F_1/F_1 g \to F_2/F_2 g \to F_3/F_3 g \to 0.$$
Since $\mathcal{F}_i|_E$ is the image of $F_i/F_i g$ in $\mathrm{coh}(E)$ for $1 \leq i \leq 3$, this proves the lemma. □

There is one further algebra that will be considered in this paper. Let $S \in \underline{\mathrm{AS}}_3'$. Write $S[g^{-1}]$ for the localization of S at the powers of g. Since g is homogeneous of degree 3, $S[g^{-1}]$ is still $\mathbb{Z}$-graded. Let

$$A = A(S) = S[g^{-1}]_0$$

denote the 0th graded piece of $S[g^{-1}]$. Just as S can be thought of as the homogeneous coordinate ring of the noncommutative projective plane qgr-S, so the algebra A can be regarded as the "coordinate ring of the quantum affine variety qgr-$S \smallsetminus E$," and one of the aims of this paper is to understand the right ideals of this ring in terms of modules in qgr-S and the geometry of $\mathbf{P}^2 \smallsetminus E$.

The next two results collect some basic facts about $A = A(S)$. Let $\Lambda_1 = S_3 g^{-1} \subset A$ and note that $\Lambda = \{\Lambda_i = \Lambda_1^i\}$ provides a filtration of A. Since g is a homogeneous normal element, $S_i g = g S_i$ for all i and so $\Lambda_i = (S_3 g^{-1})^i = S_3^i g^{-i} = S_{3i} g^{-i}$.

Lemma 2.7. *Let $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}_3'$ and set $A = A(S)$. Then:*

1. *A is generated by Λ_1 as a k-algebra.*
2. *$R = S[g^{-1}]$ is strongly graded in the sense that $R_n R_m = R_{n+m}$ for all m, n. Moreover, R is projective as a left or right A-module.*
3. *If $|\sigma| = \infty$ (as an automorphism of the scheme E) then A is a simple hereditary ring.*
4. *If $|\sigma| < \infty$ then A is an Azumaya algebra of global dimension two.*
5. *Set $\mathcal{R}(A) = \bigoplus \Lambda_1^i g^i \subset S[g^{-1}]$. Then $\mathcal{R}(A) = S^{(3)}$.*

Proof. (1) By construction, $A = \sum_{i \in \mathbf{Z}} S_{3i} g^{-i} = \sum_{i \in \mathbf{Z}} \left(S_3 g^{-1}\right)^i = \sum_{i \geq 0} \Lambda_1^i$.

(2) By [ATV2, Proposition 7.4], R is strongly graded. Thus $R_i R_{-i} = R_0 = A$ for all i and the dual basis lemma implies that each R_i is a projective A-module.

(3) A is simple by [ATV2, Theorem I]. By [NV, Theorem A.I.3.4] mod-$A \simeq$ gr-R via the map $M \mapsto M \otimes_A R$, for $M \in$ mod-A. Thus, it suffices to show that each $N \in$ gr-R has homological dimension $\mathrm{hd}(N) \leq 1$. This is proved in [Aj, Proposition 2.18] for the Sklyanin algebra and the proof works in general.

(4) A is Azumaya by [ATV2, Theorem I]. Since S has global dimension three, it follows that graded R-modules have homological dimension at most two. Thus, as in the proof of (3), $\mathrm{gldim}(A) \leq 2$. The other inequality follows from [MR, Corollary 6.2.8 and Proposition 13.10.6].

(5) By construction, $\mathcal{R}(A)$ is a graded ring with $\mathcal{R}(A)_i = \Lambda_i g^i = S_{3i} g^{-i} g^i = S_{3i}$ for all i. Therefore, $\mathcal{R}(A) = S^{(3)}$. $\square$

Corollary 2.8. *Let $S \in \underline{\mathrm{AS}}_3'$ with factor ring $B = S/gS$ and set $A = A(S)$. Then:*

1. *If g is central in S, then $S^{(3)}$ is naturally isomorphic to the Rees ring of A and $B^{(3)} \cong \mathrm{gr}_\Lambda A = \bigoplus \Lambda_i / \Lambda_{i-1}$.*
2. *If E is an integral curve then $\mathrm{gr}_\Lambda A$ is a domain.*

Proof. (1) The Rees ring of A is defined to be the graded algebra $\mathcal{R} = \bigoplus \Lambda_i t^i$, regarded as a subring of the polynomial ring $A[t]$. Since $A[t] \cong A[g] \subset S[g^{-1}]$, clearly the map $t \mapsto g$ induces an isomorphism $\mathcal{R} \cong \mathcal{R}(A) \cong S^{(3)}$. The identity $B^{(3)} \cong \mathrm{gr}_\Lambda A$ then follows from the observation that $\mathrm{gr}_\Lambda A \cong \mathcal{R}/t\mathcal{R}$.

(2) When g is not central, $\mathcal{R}(A)$ is isomorphic to a *twist* of $\mathcal{R}$, in the sense of [ATV2, Section 8]. In more detail, conjugation by g defines an automorphism τ on S and hence on A; thus $\tau(a) = gag^{-1}$. We may then define a new multiplication on $\mathcal{R}$ by $a \circ b = a\tau^r(b)$, for $a \in \mathcal{R}_r$ and $b \in \mathcal{R}_s$. It is readily checked that the map $t \mapsto g$

induces an isomorphism $(\mathcal{R}, \circ) \cong \mathcal{R}(A)$ and so $(\mathcal{R}/t\mathcal{R}, \circ) \cong \mathcal{R}(A)/g\mathcal{R}(A) = B^{(3)}$. If E is integral then B is a domain [AV] and hence so is $B^{(3)} = S^{(3)}/gS^{(3)}$. This immediately implies that $\mathcal{R}/g\mathcal{R}$ is a domain. $\square$

We end this section by describing some of the algebras appearing in the classification of $\underline{\mathrm{AS}}'_3$.

Remark 2.9. According to the classification in [ATV1, (4.13)], the algebras in $\underline{\mathrm{AS}}_3$ with qgr-$S \not\simeq \mathrm{coh}(\mathbf{P}^2)$ break into seven classes. For the reader's convenience, we note the generic example of the curve E, the automorphism σ, the number of essential parameters $\mathbf{p}$ defining the family and the defining relations $\{f_i\}$ for a typical algebra $R = k\{x, y, z\}/(f_1, f_2, f_3)$ in each class. The letters p, q, r will always denote elements of k^*.

Type A: E is elliptic with σ translation, $\mathbf{p} = 2$ and $R = \mathrm{Skl}(E, \mathcal{L}, \sigma)$.
Type B: E is elliptic with σ being multiplication by (-1) and $\mathbf{p} = 1$. Take $\{f_i\} = \{xy + yx - z^2 - y^2, x^2 + y^2 + (1-p)z^2, zx - xz + pzy - pyz\}$.
Type $\mathbf{S_1}$: E is a triangle with each component stabilized by σ and $\mathbf{p} = 3$. Take $\{f_i\} = \{xy + pyx, yz + qzy, zx + rxz\}$, with $pqr \neq -1$.
Type $\mathbf{S'_1}$: E is the union of a line and a conic, with each component stabilized by σ and $\mathbf{p} = 2$. Take $\{f_i\} = \{xy + pyx + z^2, yz + qzy, zx + qxz\}$.
Type $\mathbf{S_2}$: E is a triangle with two components interchanged by σ and $\mathbf{p} = 1$. Take $\{f_i\} = \{x^2 + y^2, yz + qzx, xz + qzy\}$.
Types $\mathbf{E}, \mathbf{H}$: E is elliptic with complex multiplication σ. Here, $\mathbf{p} = 0$.

If one takes the example R of type S_1, but with $pqr = -1$, then one obtains an algebra R with qgr-$R \cong \mathrm{coh}(\mathbf{P}^2)$.

It is useful to think of $A = A(\mathrm{Skl})$ as an "elliptic" analogue of the first Weyl algebra $A_1 = k\{u, v\}/(uv - vu - 1)$, as this will help illustrate the relationship between the results of this paper and those of Berest-Wilson [BW1, BW2] and others. The AS regular algebra associated to A_1 is

$$U = k\{x, y, z\}/(xy - yx - z^2, z \text{ central}). \tag{2.6}$$

It is easy to check that $U \in \underline{\mathrm{AS}}'_3$ (of Type S'_1), with $g = z^3$ and $A_1 \cong A(U)$.

There is one significant difference between $A(\mathrm{Skl})$ and the first Weyl algebra A_1. The group of k-algebra automorphisms $\mathrm{Aut}_k(A_1)$ is large [Di, Théorème 8.10] and this group plays an important role in the work of Berest and Wilson [BW1]. However, the opposite is true of the Sklyanin algebra.

Proposition 2.10. *Let $S = \mathrm{Skl}$ and $A = A(S)$. Write $\mathrm{PAut}_{gr}(S^{(3)})$ for the group of graded k-algebra automorphisms θ of $S^{(3)}$ satisfying $\theta(g) = g$.*

If $|\sigma| = \infty$ then $\mathrm{Aut}_k(A) \cong \mathrm{PAut}_{gr}(S^{(3)})$ is a finite group.

Since this result is not used in the paper, we will omit the proof.

3. From A-Modules to S-Modules and Back

In the commutative case, given a line bundle on $\mathbf{P}^2$, one can restrict it to $\mathbf{P}^2 \smallsetminus E$ and conversely, given a line bundle on $\mathbf{P}^2 \smallsetminus E$ one can extend it to one on $\mathbf{P}^2$. The same ideas work for qgr-S for $S \in \underline{\mathrm{AS}}'_3$. The details are given in this section, which will in particular prove that there is a (3-1) correspondence between projective right ideals of $A(\mathrm{Skl})$ and line bundles in qgr-Skl modulo shifts. We also prove analogous results for rank one torsion-free modules. This is one of a few places in

this paper where the results are distinctly different for different AS regular algebras. In particular, for arbitrary S one does not get a finite-to-one correspondence (see Proposition 3.15).

Let $S \in \underline{\mathrm{AS}}_3'$ and write $A = A(S)$. If $\mathcal{M} \in \mathrm{qgr}\text{-}S$, set $M = \Gamma^*(\mathcal{M})$ and $\mathcal{M}|_A = M|_A = M[g^{-1}]_0$. Note that if $\mathcal{M} = \pi(N)$ for some other finitely generated S-module N, then M and N only differ by a finite-dimensional S-module and so $M[g^{-1}] \cong N[g^{-1}]$. There is a natural filtration Φ^M on $M|_A$ defined by $\Phi_i^M = M_{3i}g^{-i}$. In the other direction, let $P = \bigcup \Theta_i$ be a filtered right A-module. Then $\mathcal{R}(P) = \mathcal{R}_\Theta(P) = \bigoplus_{i\geq 0} \Theta_i g^i$ is naturally a right $S^{(3)}$-module under the identification of Lemma 2.7(5). Thus, we may form $\mathcal{S}_\Theta(P) = \mathcal{R}(P) \otimes_{S^{(3)}} S$ and $\mathcal{V}_\Theta(P) = \pi\mathcal{S}(P)$. Set $R = S[g^{-1}]$ and give each graded piece $R_n = \sum S_{n+3j}g^{-j}$ the A-module filtration defined by $\Psi_j^{R_n} = S_{n+3j}g^{-j}$.

Lemma 3.1. *Let $S \in \underline{\mathrm{AS}}_3'$ and $\mathcal{M} \in \mathrm{qgr}\text{-}S$ be a torsion-free module. If $M = \Gamma^*(\mathcal{M})$ then:*

1. *$\mathcal{M} \cong \mathcal{V}(\mathcal{M}|_A)$, where $\mathcal{M}|_A$ is given the induced filtration Φ^M.*
2. *There is a filtered isomorphism $\mathcal{M}(\ell)|_A \cong (\mathcal{M}|_A) \otimes_A R_\ell$ for $\ell \in \mathbb{Z}$.*
3. *If $\mathcal{N} = \mathcal{M}g$ and $N = \Gamma^*(\mathcal{N})$ then $\mathcal{N}|_A \cong \mathcal{M}|_A$ and $\Phi_i^N = \Phi_{i-1}^M$ for $i \gg 0$.*
4. *Conversely, if $P = \bigcup \Theta_i$ is a filtered A-module then $\mathcal{S}_\Theta(P)|_A = P$.*

Proof. (1) Since M is torsion-free, $\mathcal{R}(\mathcal{M}|_A) = \bigoplus_{i\geq 0} M_{3i}g^{-1}g^i \cong M^{(3)}$. The natural map $M^{(3)} \otimes_{S^{(3)}} S \to M$ then has bounded kernel and cokernel [Ve, Theorem 4.4] and so $\mathcal{V}(\mathcal{M}|_A) \cong \mathcal{M}$.

(2) As $M|_A$ is torsion-free and R is a projective left A-module (Lemma 2.7), the natural map $(M|_A) \otimes_A R_\ell \to (M|_A)R_\ell$ is an isomorphism and we can use products in place of tensor products. The filtration on $(\mathcal{M}|_A)R_\ell$ is then the natural one defined by $\Theta_j = \sum_k \Phi_{j-k}^M \Psi_k^{R_\ell}$. Since $gS_r = S_r g$, one obtains

$$\Theta_j = \sum_k M_{3j-3k}g^{j-k}S_{3k+\ell}g^{-k} = \sum_k M_{3j-3k}S_{3k+\ell}g^{-j} = M_{3j+\ell}g^{-j},$$

for any j such that M is generated in degrees $\leq j$. On the other hand, the filtration on $M(\ell)|_A$ is given by $\Theta_j' = M(\ell)_{3j}g^{-j} = M_{\ell+3j}g^{-j}$. As these vector spaces agree for $j \gg 0$, the A-modules are also equal.

(3) This is a straightforward computation.

(4) Note that $\mathcal{S}_\Theta(P)_{3j} = \mathcal{R}_\Theta(P)_j = \Theta_j g^j$ and so $\mathcal{S}(P)|_A = \sum \Theta_j g^j g^{-j} = P$. $\square$

In order to get a way of relating unfiltered A-modules with modules in $\mathrm{Qgr}\text{-}S$ we need to relate different filtrations on A-modules. This is given by the following mild generalization of [BW2, Lemma 10.1].

Lemma 3.2. *Let $A = \bigcup_{i\geq 0} \Lambda_i$ be any filtered k-algebra (thus the Λ_i are k-subspaces of A) such that both A and $\mathrm{gr}\, A = \bigoplus \Lambda_i/\Lambda_{i-1}$ are Ore domains. Let $M = \bigcup \Theta_i$ be a filtered right ideal of A such that either $\dim_k \Theta_i < \infty$ for all i or that $\mathrm{gr}_\Theta M = \bigoplus \Theta_i/\Theta_{i-1}$ is a finitely generated $\mathrm{gr}\, A$-module.*

Assume, for some $r_0 \in \mathbb{N}$, that $(\mathrm{gr}_\Theta M)_{\geq r_0}$ is a torsion-free right $\mathrm{gr}(A)$-module. Then there exists $j \in \mathbf{Z}$ such that $\Theta_i = \Lambda_{i+j} \cap M$ for all $i \gg 0$.

Proof. We have two filtrations on M, the given filtration Θ and $\{\Gamma_i = M \cap \Lambda_i\}$. Write $\Theta_i^\circ = \Theta_i \smallsetminus \Theta_{i-1}$, and similarly for Γ and Λ. Fix $m \in \Theta_r^\circ$ and let $n \in \Theta_p^\circ$ for some $r, p \geq r_0$. If $m \in \Gamma_s^\circ$ and $n \in \Gamma_q^\circ$, we claim that $r - s = p - q$.

The assumption that $(\operatorname{gr}_\Theta M)_{\geq r_0}$ is torsion-free is equivalent to the assertion that $ma \in \Theta^\circ_{r+t}$ for all $a \in \Lambda^\circ_t$ with $t \geq 0$. As A is Ore, there exist $a, b \in A \smallsetminus \{0\}$, say $a \in \Lambda^\circ_u$ and $b \in \Lambda^\circ_v$, such that $ma = nb \neq 0$. Thus $ma \in \Theta^\circ_{r+u}$ while $nb \in \Theta^\circ_{p+v}$. So $r + u = p + v$. Since $\operatorname{gr}_\Gamma M \subseteq \operatorname{gr} A$ is a torsion-free module, the same argument implies that $ma \in \Gamma^\circ_{s+u}$ while $nb \in \Gamma^\circ_{q+v}$. Thus $s + u = q + v$. Subtracting the two equations proves the claim that $r - s = p - q$.

As n is arbitrary, the claim implies that $\Theta^\circ_p \subseteq \Gamma^\circ_{p-(r-s)}$ for all $p \geq r_0$. Conversely, either assumption on the Θ_j implies that $\Theta_{r_0} \subseteq \Gamma_{p_0}$ for some $p_0 > 0$. Pick $n' \in \Gamma^\circ_{q'}$ for some $q' > p_0$ and suppose that $n' \in \Theta^\circ_{p'}$. Necessarily, $p' \geq r_0$. Thus, applying the claim to $n = n'$ shows that $r - s = p' - q'$ and hence that $\Gamma^\circ_{q'} \subseteq \Theta^\circ_{q'+(r-s)}$. Thus, $\Theta^\circ_p = \Gamma^\circ_{p-(r-s)}$ for all $p \gg 0$. □

Notation 3.3. *We will call two filtrations Γ and Γ' on an A-module M* equivalent *if $\Gamma_i = \Gamma'_i$ for all $i \gg 0$. If M is a finitely generated torsion-free rank 1 right A-module, then we may choose an embedding $\psi : M \hookrightarrow A$ as a right ideal of A and hence obtain a filtration $\Lambda(M, \psi)$ defined by $\Lambda(M,\psi)_n = \psi^{-1}(\Lambda_n \cap \psi(M))$. By the last lemma, this filtration is unique up to shift and equivalence. We call it a* canonical filtration*.*

Let $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$ with $A = A(S)$ and $B = B(E, \mathcal{L}, \sigma) = S/gS$. We next want to determine the modules in qgr-S that appear as $\mathcal{V}_\Theta(P)$ for some $P \in$ Mod-A. Write $\mathcal{V} = \mathcal{V}_S$ for the set of isomorphism classes of rank one, torsion-free modules $\mathcal{M}$ in qgr-S for which $\mathcal{M}|_E = \mathcal{M}/\mathcal{M}g$ is a vector bundle on E. A technical but more convenient description of $\mathcal{V}_S$ follows from the next lemma.

Lemma 3.4. *Set $S = S(E, \mathcal{L}, \sigma)$ and let $\mathcal{M} \in$ qgr-S be a torsion-free module. Then $\mathcal{M}|_E$ has no simple submodules if and only if $\mathcal{M}|_E$ is a vector bundle.*

Proof. Assume that $\mathcal{M} \in$ qgr-S is a torsion-free module such that $\mathcal{M}|_E$ has no simple submodules. We claim that $\mathcal{M}$ has a resolution $0 \to \mathcal{P} \to \mathcal{Q} \to \mathcal{M} \to 0$ where $\mathcal{P}$ and $\mathcal{Q}$ are sums of shifts of $\mathcal{O}_S$. To see this, note that $M = \Gamma(\mathcal{M})$ is torsion-free and so embeds into a direct sum F of shifts of S. By Lemma 2.3(3), F/M has no finite dimensional submodules and so [ATV2, Proposition 2.46(i)] implies that $\operatorname{hd}(F/M) \leq 2$. Thus, $\operatorname{hd}(M) \leq 1$ and so M has a graded free resolution of length one. Now apply π to this resolution.

By Lemma 2.6, the complex $0 \to \mathcal{P}|_E \to \mathcal{Q}|_E \to \mathcal{M}|_E \to 0$ is also exact and so, by Lemma 2.5, $\operatorname{hd}_E(\mathcal{M}|_E) < \infty$. Since E is a Gorenstein curve and $\mathcal{M}|_E$ has no simple subobjects, $\mathcal{M}|_E$ locally has depth at least one. Thus the Auslander-Buchsbaum formula [Ei, Theorem 19.9] proves that $\operatorname{hd}_E(\mathcal{M}|_E) = 0$, locally and hence globally. Thus $\mathcal{M}|_E$ is a vector bundle. The other direction is trivial. □

Lemma 3.5. *Let $S \in \underline{\mathrm{AS}}'_3$ and set $A = A(S)$. Suppose that P is a right ideal of A with the induced filtration $\Gamma_i = P \cap \Lambda_i$. Then $\mathcal{V}(P) \in \mathcal{V}$.*

Proof. Let $Q = \mathcal{R}_\Gamma(P)$. By Lemma 2.7, Q is a right ideal of $S^{(3)}$ and hence QS is a right ideal of S. Since $(Q \otimes_{S^{(3)}} S)^{(3)} = Q = (QS)^{(3)}$, [Ve, Theorem 4.4] implies that $\mathcal{V}(P) = \pi(Q \otimes S) = \pi(QS)$ is torsion-free. Now consider $\mathcal{V}(P)|_E$. Then

$$Q/Qg \;= \bigoplus \frac{(P \cap \Lambda_i)g^i}{(P \cap \Lambda_{i-1})g^i} \cong \bigoplus \frac{(P \cap \Lambda_i + \Lambda_{i-1})g^i}{\Lambda_{i-1}g^i} \hookrightarrow S^{(3)}/gS^{(3)} = B^{(3)}.$$

Once again, [Ve] shows that $\mathcal{S}(P)/\mathcal{S}(P)g \cong (Q/Qg)B \hookrightarrow B$ in qgr-B. Thus, $\mathcal{V}(P)|_E = \pi(\mathcal{S}(P)/\mathcal{S}(P)g)$ has no simple subobjects. □

We let $\sim$ denote the equivalence relation on $\mathcal{V}$ defined by $\mathcal{M} \sim \mathcal{M}g$. Let $\mathcal{P} = \mathcal{P}_A$ denote the set of isomorphism classes of finitely generated, torsion-free rank one A-modules.

Proposition 3.6. *Let $S \in \underline{\mathrm{AS}}'_3$ and write $A = A(S)$. Then:*

(1) *The map $\mathcal{M} \mapsto \mathcal{M}|_A$ induces a surjection $\psi : (\mathcal{V}/{\sim}) \to \mathcal{P}$.*
(2) *If E is integral, then ψ is a bijection.*

Proof. (1) If $\mathcal{M} \in \mathcal{V}$, then $M = \Gamma^*(\mathcal{M})$ is torsion-free and hence so is $\mathcal{M}|_A$. By Lemma 3.1, $\mathcal{M}|_A \cong (\mathcal{M}g)|_A$ and so ψ is defined. If $P \in \mathcal{P}$, pick an embedding $P \hookrightarrow A$ and take the induced filtration $\Theta_i = P \cap \Lambda_i$. Then Lemma 3.5 ensures that $\mathcal{V}(P) \in \mathcal{V}$ and so ψ is surjective.

(2) In this case, Corollary 2.8 implies that $\mathrm{gr}_\Lambda A$ is a domain and hence Lemma 3.2 ensures that the filtration Θ in part (1) is unique up to shift and equivalence. Given two equivalent filtrations, say Θ and Θ' of P, then $\mathcal{S}_\Theta(P)$ and $\mathcal{S}_{\Theta'}(P)$ only differ in a finite number of graded pieces and so $\mathcal{V}_\Theta(P) = \mathcal{V}_{\Theta'}(P)$. On the other hand, suppose that $\Theta'_i = \Theta_{i-1}$, for $i \gg 0$. Then, in high degree, $\mathcal{R}_{\Theta'}(P) = \sum \Theta_{i-1}g^i = \mathcal{R}_\Theta(P)g$. Thus, $\mathcal{V}_\Theta(P) = \mathcal{V}_{\Theta'}(P)g$ and so $\mathcal{V}_\Theta(P) \sim \mathcal{V}_{\Theta'}(P)$. Thus, ψ is a bijection. $\square$

We want to refine Proposition 3.6(2) in two ways. In Theorem 3.8 we will use an analogue of the first Chern class to chose a canonical member of each equivalence class of $\sim$. Then we will show that under this equivalence the projective $A(S)$-modules correspond precisely to the line bundles in qgr-S (see Corollary 3.12).

Let $\mathcal{M} \in$ qgr-S for $S \in \underline{\mathrm{AS}}_3$. The *first Chern class* $c_1(\mathcal{M})$ of $\mathcal{M}$ is defined by the properties that it should be additive on short exact sequences and satisfy $c_1(\mathcal{O}_S(j)) = j$. As the next lemma shows, this uniquely determines $c_1(\mathcal{M})$: one simply takes a resolution of $\mathcal{M}$ by shifts of $\mathcal{O}_S$ and applies additivity. When qgr-$S \simeq \mathrm{coh}(\mathbf{P}^2)$, it is easy to see that this definition of c_1 coincides with the usual commutative one.

We first describe some of the basic properties of $c_1(\mathcal{M})$.

Lemma 3.7. *Let $\mathcal{M} \in$ qgr-S for $S \in \underline{\mathrm{AS}}_3$.*

(1) *There is a unique function $\mathcal{M} \mapsto c_1(\mathcal{M})$ with the given properties.*
(2) *$c_1(\mathcal{M}(s)) = c_1(\mathcal{M}) + s \cdot \mathrm{rk}(\mathcal{M})$ for any $s \in \mathbb{Z}$.*
(3) *Assume that $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$ for a smooth curve E and that $\mathcal{M}$ is torsion-free. Then $c_1(\mathcal{M}) = \frac{1}{3}\deg(\mathcal{M}|_E)$.*

Proof. (1) Let $M \in$ gr-S. For $q \geq 0$, write $\mathrm{Tor}_q^S(M, k) = \bigoplus_j k(\ell_{qj})$, as a graded k-module and define $c'_1(M) = \sum_{q,j}(-1)^q \ell_{qj}$. Clearly, $c'_1(-)$ is additive on short exact sequences and is well defined. Thus it suffices to prove that $c'_1(M) = c_1(\pi(M))$.

If M has a graded projective resolution $P^\bullet \to M \to 0$, with $P^q = \bigoplus_j S(m_{qj})$, then, as in [ATV1, (2.8)], $c'_1(M) = \sum_{q,j}(-1)^q m_{qj}$. On the other hand, [ATV1, (2.15)] implies that $c'_1(k) = 0$ and hence that $c'_1(F) = 0$ for any finite dimensional graded S-module F. Thus, Lemma 2.3(3) implies that $c'_1(M) = c'_1(\Gamma^*(\pi(M)))$, for any $M \in$ gr-S. Thus, defining $c_1(\pi(M)) = c'_1(M)$ does indeed give a unique, well-defined function determined by the properties that it is additive on exact sequences and satisfies $c_1(\mathcal{O}_S(m)) = m$ for $m \in \mathbb{Z}$.

(2) By additivity of c_1 and rank, it suffices to prove this for $\mathcal{M} = \mathcal{O}_S(t)$, for which it is obvious.

(3) If $P^\bullet \to \mathcal{M} \to 0$ is a resolution of $\mathcal{M}$ by shifts of $\mathcal{O}_S$, then Lemma 2.6 implies that $P|_E^\bullet \to \mathcal{M}|_E \to 0$ is a resolution of $\mathcal{M}|_E$ by shifts of $\mathcal{O}_E$. By Lemma 2.5, $\deg \mathcal{O}(n)|_E = 3n$, for any n and so the result follows from additivity. $\square$

Theorem 3.8. *Let $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}_3'$ be such that E is integral. Then:*

1. *There is a bijection between $\mathcal{P}$ and $\{\mathcal{M} \in \mathcal{V} : 0 \le c_1(\mathcal{M}) \le 2\}$.*
2. *There exists a (3-1) correspondence between modules $P \in \mathcal{P}$ and modules $\mathcal{M} \in \mathcal{V}$ satisfying $c_1(\mathcal{M}) = 0$.*

Remark 3.9. (1) By Remark 2.9, this theorem applies to algebras of types **A, B, E** and **H**. We give a brief discussion of the other cases at the end of the section.

(2) This result proves Theorem 1.5 from the introduction, modulo a proof of Theorem 1.1(2).

Proof. (1) Suppose that $M \in \text{gr-}S$ has a projective resolution $P^\bullet \to M \to 0$. Since $S(n)g = gS(n) \cong S(n-3)$, the module Mg has a projective resolution $P^\bullet g \cong P^\bullet(-3)$. Thus, by Lemma 3.7(2), $c_1(\mathcal{M}g) = c_1(\mathcal{M}) - 3$ for any torsion-free, rank one module $\mathcal{M} \in \text{qgr-}S$. Now apply Proposition 3.6(2).

(2) Let $\mathcal{M} \in \mathcal{V}$ be such that $0 \le \mathcal{M} \le 2$. By Lemma 3.7(2), again, there exists a unique $r \in \{0, 1, 2\}$ such that $c_1(\mathcal{M}(-r)) = 0$. By Lemma 3.1(2), $\mathcal{M}(-r)|_A \cong (\mathcal{M}|_A) \otimes R_{-r}$, giving the (3-1) equivalence. $\square$

A natural question raised by Theorem 3.8 is whether one can identify projective right ideals of A in terms of qgr-S-modules. As we show, they correspond precisely to line bundles in qgr-S, as defined below.

If $S \in \underline{\mathrm{AS}}_3$ and $\mathcal{M} \in \text{qgr-}S$, write

$$(3.1) \qquad \underline{\mathrm{Hom}}_{\text{qgr-}S}(\mathcal{M}, \mathcal{O}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\text{qgr-}S}(\mathcal{M}, \mathcal{O}(n)) \in S\text{-gr}.$$

One can think of $\pi\underline{\mathrm{Hom}}$ as sheaf Hom on qgr-S. The right derived functors of $\pi\underline{\mathrm{Hom}}$ will be denoted by $\pi\underline{\mathrm{Ext}}$ and these are again objects in S-qgr. A more complete discussion of these concepts can, for example, be found in [KKO, Section 5.3]. In particular, it is observed there that $\pi\underline{\mathrm{Hom}}_{\text{qgr-}S}(\mathcal{M}, \mathcal{O}) \cong \pi\underline{\mathrm{Hom}}_{\text{gr-}S}(\Gamma^*(M), S)$, and so this notation is consistent with the earlier definitions of $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Ext}}$.

Definition 3.10. ([KKO, Definition 5.4]) Let $S \in \underline{\mathrm{AS}}_3$. An object $\mathcal{F} \in \text{qgr-}S$ is called a *vector bundle* if $\pi\underline{\mathrm{Ext}}^i(\mathcal{F}, \mathcal{O}) = 0$ for all $i > 0$. Such an object is necessarily torsion-free. If $\mathcal{F}$ is a vector bundle of rank one, then $\mathcal{F}$ is called a *line bundle*.

This definition of vector bundle is only appropriate for rings of finite homological dimension that satisfy the χ condition. In particular, the analogous definition for qgr-S/gS need not correspond to vector bundles in $\mathrm{coh}(E)$ and so we do not talk about vector bundles in qgr-S/gS.

Lemma 3.11. *Let $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}_3'$ and $\mathcal{M} \in \text{qgr-}S$. Then:*

1. *$\mathcal{M}$ is a vector bundle if and only if $\Gamma(\mathcal{M})$ is a reflexive S-module.*
2. *If $\mathcal{M}$ is a vector bundle, then $\mathcal{M}|_E$ is a vector bundle over E.*

Proof. (1) This follows from the Auslander-Gorenstein and Cohen-Macaulay conditions (see [ATV2, Note, p.352]). In more detail, let $\mathcal{M} \in \text{qgr-}S$ be a torsion-free module. Since $\mathcal{M}$ embeds in a direct sum of shifts of $\mathcal{O}_S$, Lemma 2.3(4) implies that $M = \Gamma(\mathcal{M})$ is finitely generated. Set $M^* = \underline{\mathrm{Hom}}(M, S)$. By [ATV2, Theorem 4.1], $\dim_k \underline{\mathrm{Ext}}^j(M, S) < \infty$ for $j > 1$ and GKdim $\underline{\mathrm{Ext}}^1(M, S) \le 1$. By [ATV2,

(4.4)] the canonical map $M \to M^{**}$ has cokernel $Q \subseteq \underline{\mathrm{Ext}}^2(\underline{\mathrm{Ext}}^1(M, S), S)$. By Lemma 2.3(3), M has no finite dimensional extensions, so M is reflexive if and only if $\dim_k Q < \infty$. By [ATV2, Theorem 4.1(iii)] this happens if and only if $\dim_k \underline{\mathrm{Ext}}^1(M, S) < \infty$.

By [KKO, Equation 14, p. 406], $\pi\underline{\mathrm{Ext}}^j(\mathcal{M}, \mathcal{O}) = \pi(\underline{\mathrm{Ext}}^j(\Gamma^*(\mathcal{M}), S))$. Thus, $\mathcal{M}$ is a vector bundle if and only if $\dim_k \underline{\mathrm{Ext}}^j(\Gamma^*(\mathcal{M}), S) < \infty$ for all $j > 0$. By the last paragraph this is equivalent to M being reflexive.

(2) Let $M = \Gamma(\mathcal{M})$ and $N = M^*$. Since M is reflexive and hence torsion-free, it is easy to see that $M/Mg \hookrightarrow \mathrm{Hom}_S(N, S/gS) = \mathrm{Hom}_{S/gS}(N/gN, S/gS)$. Therefore, $\mathcal{M}|_E = \pi(M/Mg) \subset \pi(\mathrm{Hom}_{S/gS}(N/gN, S/gS))$ which certainly has no simple submodules. Now apply Lemma 3.4. □

Corollary 3.12. *Let $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$ with $A = A(S)$ and let $\mathcal{M} \in \text{qgr-}S$ be such that $\mathcal{M}|_E$ is a vector bundle. Then:*

(1) *$\mathcal{M}$ is a vector bundle if and only if $\mathcal{M}|_A$ is a projective A-module.*

(2) *If $|\sigma| = \infty$ as an automorphism of the scheme E, then $\mathcal{M}$ is automatically a vector bundle.*

Proof. (1) Set $M = \Gamma(\mathcal{M})$, with double dual M^{**}. By [ATV2, Corollary 4.2(iv)], $\mathrm{GKdim}(M^{**}/M) \leq 1$. Suppose, first, that there is a proper essential extension $0 \to M \to N \to H \to 0$ with $\mathrm{GKdim}(H) \leq 1$ and $Hg = 0$. Since M has no extensions by finite dimensional modules, $\mathrm{GKdim}\, H = 1$. Consider the exact sequence

$$\mathrm{Tor}^1_S(H, S/Sg) \xrightarrow{\theta} M/Mg \longrightarrow N/Ng \longrightarrow H/Hg \longrightarrow 0.$$

Now, $\mathrm{GKdim}\,\mathrm{Tor}^1_S(H, S/Sg) \leq \mathrm{GKdim}\, H \leq 1$ by [ATV2, Proposition 2.29(iv)]. By Lemma 3.4, this forces $\mathrm{Im}(\theta) = 0$ and hence $Mg = M \cap Ng$. On the other hand, $Ng \subseteq M$ since $Hg = 0$. Thus, $Mg = Ng$. Since M and N are torsion-free, this implies that $M = N$. Thus, no such extension exists.

By Lemma 3.11, $\mathcal{M}$ is a vector bundle if and only if M is reflexive. By the last paragraph this happens if and only if $M[g^{-1}]$ is reflexive. By the equivalence of categories $\text{mod-}A \simeq \text{gr-}S[g^{-1}]$ [NV, Theorem A.I.3.4], this happens if and only if $\mathcal{M}|_A = M[g^{-1}]_0$ is reflexive. Finally, Lemma 2.7 implies that $\mathrm{gldim}(A) \leq 2$ and so reflexive A-modules are the same as projective A-modules.

(2) This is immediate from part (1) combined with Lemma 2.7(3). □

Remark 3.13. Despite the fact that it holds for all values of $|\sigma|$, there is actually a striking dichotomy in Theorem 3.8: If $|\sigma| = \infty$ then Corollary 3.12(2) applies. In contrast, when $|\sigma| < \infty$, A has global (and Krull) dimension two. It follows that A has many non-projective right ideals and hence that $\mathcal{V}$ contains many modules $\mathcal{M}$ that are not line bundles.

The reason for demanding that E be integral in Proposition 3.6(2) and Theorem 3.8 is so that Lemma 3.2 can be applied to restrict the number of possible filtrations on an A-module P. If E is not integral, then that lemma can fail and so the map ψ in Proposition 3.6(1) need not be a bijection.

An example where ψ has infinite fibres is given by the ring

$$T = k\{x, y, z\}/(yx - pxy, xz - zx, zy - yz), \tag{3.2}$$

where $p \in k$ is transcendental over the prime subfield. In the notation of Remark 2.9, T is an AS-regular ring of Type $\mathbf{S}_1$ with $g = xyz$. Moreover, T is an

Ore extension $T = R[x,\tau]$, where τ is an automorphism of the polynomial ring $R = k[y,z]$; thus multiplication is defined by $xr = \tau(r)x$ for $r \in R$. We regard elements of T as polynomials in x with coefficients in R and write $\deg_x$ for the corresponding degree function.

A method for constructing non-free projective modules over Ore extensions is given in [St1, Theorem 1.2] and we use a similar technique to build reflexive T-modules.

Lemma 3.14. *Define T by (3.2) and set $M = \{t \in T : (z+y)t \in (x+z)T\}$. Then:*

1. *M contains $\alpha = x^2 + x(1+p)z + pz^2$ and $\beta = x(z+y) + z(z+py)$.*
2. *M contains no polynomial of the form $\gamma = xz^i + r$ for $r \in R$ and $i \geq 0$.*
3. *$M[g^{-1}]_0$ is a non-cyclic projective right ideal of $A(T)$.*

Proof. (1) Use the two identities $(y+z)\alpha = (x+z)\big(x(p^2y+z) + pz(y+z)\big)$ and $(y+z)\beta = (x+z)(y+z)(py+z)$.

(2) Suppose that $xz^i + r \in M$, for some $i \geq 0$ and $r \in R$. Then

$$(z+y)(xz^i + r) = x(z^{i+1} + pyz^i) + (z+y)r = (x+z)(z^{i+1} + pyz^i) + w,$$

for $w = (z+y)r - (z^{i+1} + pyz^i)$. Since $\deg_x w = 0$, the definition of M forces $w = 0$. But the equation $w = 0$ is impossible in the polynomial ring $k[y,z]$.

(3) This follows easily from [St1, Theorem 1.2]. □

Extend τ to an automorphism of T by defining $\tau(t) = xtx^{-1}$, for $t \in T$.

Proposition 3.15. *Let M be the T-module defined in Lemma 3.14. Then:*

1. *$\pi(M)$ is a line bundle such that $\pi(M) \ncong \pi(M^{\tau^i}[j])$, for $(i,j) \neq (0,0)$.*
2. *However, $M[g^{-1}]_0 \cong N[g^{-1}]_0$, where $N = M^{\tau^i}[j]$, for any $i, j \in \mathbb{Z}$.*
3. *Thus, the map ψ defined in Proposition 3.6(1) has some infinite fibres.*

Proof. (1) Since M is reflexive, Lemma 3.11 implies that $\pi(M)$ is a line bundle.

Suppose there exists an isomorphism $\theta : \pi(M) \to \pi(N)$, where $N = \pi(M^{\tau^i}[j])$, for some i, j. Then θ induces a homomorphism θ from M to the injective hull of N such that $\theta(M_{\geq n}) = N_{\geq n}$, for all $n \gg 0$. However, by construction, M and N are reflexive and so have no essential extensions by finite dimensional T-modules. Thus, θ induces an isomorphism from M to N. Since τ is a graded isomorphism of T, this implies that M and N have the same Hilbert series. As M and $M^{\tau^i}[j]$ have different Hilbert series for $j \neq 0$, this forces $j = 0$.

We may write $f_1M = f_2N$ for some $f_i \in T_d$ and some d. By [RSS], elements in T that are monic when regarded as polynomials in x form an Ore set $\mathcal{C}$. Localizing at $\mathcal{C}$ and using Lemma 3.14(1) gives the identity $M_{\mathcal{C}} = T_{\mathcal{C}} = N_{\mathcal{C}}$. Thus $f_1T_{\mathcal{C}} = f_2T_{\mathcal{C}}$. The only units in $T_{\mathcal{C}}$ are of the form $\lambda_1 g_1^{-1}g_2$, where $\lambda \in k^*$ and $g_i \in \mathcal{C}$. Thus, we may assume that the $f_i \in \mathcal{C}$.

Consider elements of least degree in x in $f_1M = f_2N$. By Lemma 3.14, this degree is $(d+1)$ and $f_2N \ni f_2\beta^{\tau^i} = x^{d+1}(z + p^{-i}y) +$ lower order terms. But, $f_1M \ni f_1\beta$, with leading term $x^{d+1}(z+y)$. Thus $f_2\beta^{\tau^i} - p^{-i}f_1\beta \in f_1M$ has leading term $x^{d+1}z(1-p^{-i})$. Since $p^i \neq 1$, this implies that M contains an element of degree 1 with leading term xz. This contradicts Lemma 3.14(2).

(2) Both $w = x$ and $w = z$, are normal elements in T. Thus, for any right ideal M' of T there exists a short exact sequence: $0 \to M'w \to M' \to M'/M'w \to 0$. The map $\psi : P \mapsto P[g^{-1}]_0$ is exact and so $\psi(M') = \psi(M'w)$. If $w = z$, then z is

central and so $M'z = zM' \cong M'[-1]$. By induction, $\psi(M) = \psi(M[i])$ for all $i \in \mathbb{Z}$. On the other hand, $M'x = x\tau^{-1}(M') = \tau^{-1}(M')[-1]$. Thus a second induction proves (2).

(3) This is immediate from (1) and (2). □

One reason why Theorem 3.8 fails for T (and other examples where E is not integral) is that g is a product of normal elements. This introduces extra units into $A(S)$ and "too many" isomorphisms between projective modules. However, if one is willing to change the ring $A(S)$ then one can still get an analogue of Theorem 3.8. We end this section by outlining the result but since it depends upon a case by case analysis the details will be left to the reader.

Let $S \in \underline{\mathrm{AS}}'_3$ and suppose first that E has an integral component X of degree $r \leq 2$ that is fixed by σ. Using the arguments of [AS, Section 4] one can show that $B' = B(X, \mathcal{L}|X, \sigma|_X) = S/hS$ for some normal element $h \in S_r$ that divides g. This happens, for example, in Types $\mathbf{S}_1$ (where $r = 1$) and $\mathbf{S}'_1$ (where r can be 1 or 2). In Type $\mathbf{S}_2$ it only happens when X is the line fixed by σ. Now set $A' = S[h^{-1}]_0$; thus $A(S)$ is a localization of A'. Since X is integral, B' is a domain and the argument of Corollary 2.8(2) shows that $\text{gr-}_{\Lambda'}A'$ is also a domain, where $\Lambda'_i = S_{ir}h^{-i}$. The proof of Theorem 3.8 then goes through to give an r-to-1 correspondence between isomorphism classes of rank one torsion-free A'-modules and elements of $\mathcal{V}$ with $c_1 = 0$. When $S = U$ is the homogenized Weyl algebra, $g = h^3$ and so $A' = A(S)$ and one recovers the result from [BW1].

There are two further cases to be considered. First consider S of type $\mathbf{S}_2$ and let Y be the union of the lines interchanged by σ. Then $B'' = B(Y, \mathcal{L}|_Y, \sigma|_Y) = S/h'S$ where $\deg h' = 2$. In this case B'' is a prime ring that is not a domain, so the analysis of the last paragraph will fail. However, if one replaces S by the Veronese ring $S^{(2)}$ then h' will become a product of two normal elements (this is because σ^2 fixes the components of Y) and the above analysis can be pushed through. The final case is when E is a product of 3 lines cyclicly permuted by σ (this is a degenerate Type $\mathbf{A}$ example, when $b = 0$ in (2.2)). Here one has to use the 3-Veronese ring $S^{(3)}$.

4. Cohomology and Base Change in Noncommutative Geometry

In this section we describe two cohomological results that will be needed in the sequel. Both are minor variants of results from the literature. The first describes the general "Cohomology and Base Change" machinery from [EGA] in the appropriate generality to apply to noncommutative projective schemes. These theorems give detailed information about the variation of cohomology in a family parametrized by a noetherian base scheme U. The second result describes the noncommutative Čech cohomology of [VW1] in a form appropriate to our needs.

The results of this section need rather strong hypotheses, but these are probably necessary. However, as will be shown in Lemma 5.1 they do hold for $S_A = S \otimes_k A$, where A is a commutative noetherian k-algebra and $S \in \underline{\mathrm{AS}}_3$.

4.1. **Cohomology and Base Change.** The results of this subsection will require the following hypotheses on our algebras:

Hypotheses 4.1. *Fix a commutative noetherian ring A and a finitely generated, connected graded A-algebra S. Assume that:*

1. *S is* strongly noetherian *in the sense that $S_B = S \otimes_A B$ is noetherian for every commutative noetherian A-algebra B.*

(2) *S satisfies χ and* Qgr-S *has finite cohomological dimension.*

If B is a commutative A-algebra and S is a cg A-algebra, we regard $S_B = S \otimes_A B$ as a cg B-algebra. By [AZ2, Proposition B8.1], Qgr-S_B is equivalent to the category of B-objects in Qgr-S. If $\mathcal{M} \in$ qgr-S_B, with S_B noetherian, then [AZ2, Lemma C6.6] implies that there is a canonical identification of cohomology groups $\mathrm{H}^i(\text{Qgr-}S_B, \mathcal{M}) = \mathrm{H}^i(\text{Qgr-}S, \mathcal{M})$ via the natural functor Qgr-$S_B \to$ Qgr-S and so we may write this group as $\mathrm{H}^i(\mathcal{M})$ without confusion. Generalizing earlier notation, set $\mathcal{O}_{S_B} = \pi(S_B) \in$ qgr-S_B. If the context is clear, we will typically write $\mathcal{O}$ for $\mathcal{O}_{S_B}$. An object $\mathcal{M} \in$ Qgr-S is *A-flat* if the functor $- \otimes_A \mathcal{M} : \text{Mod-}A \longrightarrow \text{Qgr-}S$ is exact. When S_A is strongly noetherian, [AZ2, Lemma E5.3] implies that $\mathcal{M} \in$ qgr-S is A-flat if and only if $\mathrm{H}^0(\text{qgr-}S, \mathcal{M}(n))$ is flat for $n \gg 0$.

Theorem 4.2 (Theorem on Formal Functions)**.** *Assume that A and S satisfy Hypotheses 4.1 and let $\mathcal{F}, \mathcal{G} \in$* qgr-$S$*. Then for every i and every ideal $\mathfrak{m}$ of A, the canonical homomorphism*

$$\mathrm{Ext}^i_{\text{qgr-}S}(\mathcal{F}, \mathcal{G}) \otimes \widehat{A} \longrightarrow \varprojlim \mathrm{Ext}^i_{\text{qgr-}S}(\mathcal{F}, \mathcal{G} \otimes_A A/\mathfrak{m}^k)$$

is an isomorphism.

Proof. [AZ2, Proposition C6.10(i)] implies that S satisfies the strong χ condition of [AZ2, (C6.8)], while [AZ2, Proposition C6.9] implies that Qgr-S is Ext-finite. Thus the hypotheses of [AZ2, Theorem D5.1] (with $k = A$ and $R = \widehat{A}$) are satisfied. □

We remark that, for a commutative ring R, a "point y of $\operatorname{Spec} R$" means any point (not necessarily a closed point).

Theorem 4.3 (Cohomology and Base Change)**.** *Assume that A and S satisfy Hypotheses 4.1 and let $y \in \operatorname{Spec} A$. Pick $\mathcal{F}, \mathcal{G} \in$* qgr-$S$ *such that $\mathcal{G}$ is A-flat. Then:*

(1) *If the natural map*
$$\phi_i : \mathrm{Ext}^i(\mathcal{F}, \mathcal{G}) \otimes_A k(y) \longrightarrow \mathrm{Ext}^i(\mathcal{F}, \mathcal{G} \otimes_A k(y))$$
is surjective, then it is an isomorphism.
(2) *If ϕ_{i-1} and ϕ_i are surjective, then $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ is a vector bundle in a neighborhood of y in* $\operatorname{Spec} A$.
(3) *If $\mathrm{Ext}^{i+1}(\mathcal{F}, \mathcal{G} \otimes k(y)) = \mathrm{Ext}^{i-1}(\mathcal{F}, \mathcal{G} \otimes k(y)) = 0$, then $\mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$ is a vector bundle in a neighborhood of y in* $\operatorname{Spec} A$ *and ϕ_i is an isomorphism.*
(4) *If $\mathcal{F}$ and $\mathcal{O}_S$ are also A-flat, then*
$$\mathrm{Ext}^i_{\text{qgr-}S}(\mathcal{F}, \mathcal{G} \otimes k(y)) = \mathrm{Ext}^i_{\text{qgr-}S \otimes k(y)}(\mathcal{F} \otimes k(y), \mathcal{G} \otimes k(y)).$$

Proof. Since $\mathcal{G}$ is A-flat, the collection of functors $M \mapsto T^i(M) = \mathrm{Ext}^i(\mathcal{F}, \mathcal{G} \otimes_A M)$ forms a cohomological δ-functor in the sense of [EGA, Section III.7] or [Ha, Section III.1]. By our assumptions, $T^i(M) \in$ mod-A for every $M \in$ mod-A. Moreover, T^i commutes with colimits and, by Theorem 4.2, the canonical homomorphism $T^i(M)^\wedge \longrightarrow \varprojlim T^i(M \otimes A/I^k)$ is an isomorphism for every $M \in$ mod-A and ideal $I \subset A$. The proof of [Ha, Proposition 12.10] now shows that, if ϕ_i is surjective, then T^i is a right exact functor of A_P-modules, where P is the ideal of A corresponding to y. Thus ϕ_i is an isomorphism by [EGA, Proposition III.7.3.1], proving (1).

If ϕ_{i-1} is also surjective then [EGA, Proposition III.7.5.4] implies that $T^i(A_P)$ is a free A_P-module. Since T^i commutes with colimits we have $T^i(A)_P = T^i(A_P)$,

proving (2). Finally, part (3) follows from [EGA, Corollaire III.7.5.5] and part (4) from [AZ2, Proposition C3.4(i)]. □

Remark 4.4. One consequence of Theorem 4.3(4) is that, under the hypotheses of that result, $\mathrm{H}^i(\mathrm{Qgr}\text{-}S, \mathcal{G}\otimes k(y)) = \mathrm{H}^i(\mathrm{Qgr}\text{-}S\otimes_A k(p), \mathcal{G}\otimes k(y))$, and so we can write this group as $\mathrm{H}^i(\mathcal{G}\otimes k(y))$ without ambiguity.

4.2. **Schematic Algebras and the Čech Complex.** Let A be a commutative ring and S a noetherian connected graded A-algebra. Following [VW1], we say S is *A-schematic* if there is a finite set $C_1, \dots, C_N$ of two-sided homogeneous Ore sets satisfying

(1) $C_i \cap S_+ \neq \emptyset$ for $i = 1, \dots, N$.
(2) For all $(c_1, \dots, c_N) \in \prod_{i=1}^N C_i$ there exists $m \in \mathbf{N}$ with $(S_+)^m \subseteq \sum_{i=1}^N c_i \cdot S$.

Given an A-schematic algebra S, fix Ore sets $C_1, \dots, C_N$ as in the definition and set $I = \{1, 2, \dots, N\}$. Given $w = (i_0, \dots, i_{p-1}) \in I^p$, set $Q_w = S_{C_{i_0}} \otimes_S \cdots \otimes_S S_{C_{i_{p-1}}}$, the tensor product of localizations. The reader should be warned that the various localizations will not commute in general and so Q_w need not be a ring and it does depend on the ordering of w. Of course Q_w is an S-bimodule. The *noncommutative Čech complex* $\mathbf{C}^\bullet$ is then given by setting

$$\mathbf{C}^p = \prod_{w\in I^{p+1}} Q_w, \quad p \geq 0,$$

and taking as differentials $\mathbf{C}^p \to \mathbf{C}^{p+1}$ the usual alternating sum of maps just as in the commutative setting.

Define a functor from Gr-S to complexes of graded S-modules by $M \mapsto M \otimes_S \mathbf{C}^\bullet$. Note that, since $C_j \cap S_+ \neq \emptyset$ for every j, if T is a torsion module then $T \otimes_S \mathbf{C}^\bullet = 0$. Hence $- \otimes_S \mathbf{C}^\bullet$ descends to a functor from Qgr-S to the category of complexes of graded S-modules. We write $\check{\mathbf{H}}^i(\mathcal{M}) = \mathrm{H}^i(\mathcal{M} \otimes_S \mathbf{C}^\bullet)$ for $\mathcal{M} \in$ Qgr-S. This is a graded S-module and we define $\check{H}^i(\mathcal{M}) = \mathrm{H}^i(\mathcal{M} \otimes_S \mathbf{C}^\bullet)_0$; equivalently $\check{H}^i(\mathcal{M})$ is the cohomology of $\mathcal{M} \otimes \mathbf{C}^\bullet_0$.

Given $\mathcal{M} \in$ Qgr-S, write $\mathbf{H}^i(\mathcal{M}) = \bigoplus_{n\in\mathbf{Z}} \mathrm{H}^i(\mathrm{Qgr}\text{-}S, \mathcal{M}(n))$. This is naturally a graded S-module with $\mathbf{H}^0(\mathcal{M})$ being the module $\Gamma(\mathcal{M})$ defined after (2.4).

Theorem 4.5 ([VW1])**.** *Suppose that S is a noetherian A-schematic algebra. Then there is an isomorphism of δ-functors $\mathbf{H}^i \to \mathrm{H}^i(- \otimes_S \mathbf{C}^\bullet)$.*

Moreover, this isomorphism is graded in the sense that $\mathrm{H}^i(\mathrm{Qgr}\text{-}S_A, \mathcal{M}(j)) \cong \check{H}^i(\mathcal{M}(j))$ for all i and j.

Proof. The first assertion of the theorem is proved in [VW1, Theorem 4] when A is a field, but the proof works equally well in the more general context.

The final claim of the theorem is only implicit in [VW1] but since the proof is short we will give it. It suffices to take $j = 0$. Recall that the Q_w are flat S-modules. Thus, for any short exact sequence $0 \to M \to N \to P \to 0$ in Gr-S one obtains an exact sequence

$$0 \to \mathbf{C}^p(M)_0 \to \mathbf{C}^p(N)_0 \to \mathbf{C}^p(P)_0 \to 0.$$

It follows that the homology functors $\check{\mathbf{H}}^i_0 = \check{H}^i$ form a cohomological δ-functor. If $E \in$ Gr-S is injective, then [VW1, Theorem 3] implies that $\check{\mathbf{H}}^i(E) = 0 = \check{\mathrm{H}}^i(E)$ for all $i > 0$. Thus the δ-functor is effaceable. Finally, as is observed in [VW1,

p.79], $\check{\mathbf{H}}^0(\mathcal{M}) = \Gamma^*(\mathcal{M})$ and so $\check{\mathrm{H}}^0(\mathcal{M}) = \mathrm{Hom}_{\mathrm{Qgr}\text{-}S}(\pi(S), \mathcal{M}) = \mathrm{H}^0(\mathcal{M})$ for any $\mathcal{M} \in \mathrm{Qgr}\text{-}S$. Therefore the hypotheses of [Ha, Corollary III.1.4] are satisfied and $\mathrm{H}^i_{\mathrm{Qgr}\text{-}S}(\mathcal{M}) \cong \check{H}^i(\mathcal{M})$ for all i. □

Remark 4.6. We have already noted that the results in this section apply to S_A, where A is a commutative noetherian k-algebra and $S \in \underline{\mathrm{AS}}_3$. However, they do not apply to all cg noetherian algebras. Specifically the noetherian cg algebras studied in [Rg] are not strongly noetherian, do not satisfy χ and are not schematic (see [Rg, Theorems 1.2, 1.3 and Proposition 11.8], respectively).

One can use the noncommutative Čech complex to prove analogues of Theorem 4.3 under formally weaker hypotheses than those used here, but we know of no applications of those generalizations.

5. Monads and the Beilinson Spectral Sequence

A classical approach to the study of moduli of sheaves on $\mathbf{P}^2$ uses the Beilinson spectral sequence to describe sheaves as the homology of monads. This can then be used to reduce the moduli problem to a question in linear algebra. In this section we show that an analogue of the Beilinson spectral sequence also works for vector bundles on a noncommutative $\mathbf{P}^2$; in other words for vector bundles in qgr-S when $S \in \underline{\mathrm{AS}}_3$. As will be seen in the later sections, this will enable us to construct a projective moduli space as a GIT quotient of a subvariety of a product of Grassmannians.

The following properties of S will be needed. We use the standard definition of a Koszul algebra as given, for example, in [KKO, Definition 4.4].

Lemma 5.1. *If $S \in \underline{\mathrm{AS}}_3$, then S is strongly noetherian, schematic and Koszul.*

Proof. That S is Koszul follows from [Sm, Theorem 5.11]. By [ATV1, Theorem 2], either S or a factor ring S/gS is a twisted homogeneous coordinate ring. In either case, by [ASZ, Propositions 4.9(1) and 4.13], S is strongly noetherian.

The fact that the Sklyanin algebra is schematic is proved in detail in [VW2, Theorem 4]. The idea of the proof is that one can reduce to the case when the base field k is a finite field. The algebra is then a PI algebra, for which the result is easy. As in the proof of [ATV2, Theorem 7.1], this argument works in general. □

Definition 5.2. Let $S \in \underline{\mathrm{AS}}_3$. A complex $\mathbf{K}$ in qgr-S_R is called a *monad* if it has the form

$$\mathbf{K}: \quad 0 \to V_{-1} \otimes_R \mathcal{O}(-1) \xrightarrow{A_{\mathbf{K}}} V_0 \otimes_R \mathcal{O} \xrightarrow{B_{\mathbf{K}}} V_1 \otimes_R \mathcal{O}(1) \to 0 \tag{5.1}$$

with the following properties:

(1) The V_i are finitely generated projective R-modules.
(2) For every $p \in \operatorname{Spec} R$, the complex $\mathbf{K} \otimes_R k(p)$ is exact at $V_{-1} \otimes k(p) \otimes \mathcal{O}(-1)$ and $V_1 \otimes k(p) \otimes \mathcal{O}(1)$.

Let $\mathrm{Monad}(S_R)$ denote the category of monads, with morphisms being homomorphisms of complexes. A monad $\mathbf{K}$ is called *torsion-free* if the cohomology of the complex $\mathbf{K} \otimes_R k(p)$ at $V_0 \otimes k(p) \otimes \mathcal{O}$ is torsion-free for all $p \in \operatorname{Spec} R$.

The terms in the monad $\mathbf{K}$ are indexed so that $V_i \otimes \mathcal{O}(i)$ lies in cohomological degree i. Thus one has a functor $\mathrm{H}^0 : \mathrm{Monad}(S_R) \to \mathrm{qgr}\text{-}S_R$ given by $\mathbf{K} \mapsto \mathrm{H}^0(\mathbf{K})$.

It is convenient through much of the paper to have a general definition for a property P of objects of qgr-S to apply to a family:

Definition 5.3. Assume that $\mathbf{K}$ is an object or a complex in qgr-S_R. Let P be a property of objects or complexes of qgr-S (for example, P could be "torsion-free," or "(semi)stable" (page 30)). We say that $\mathbf{K}$ *has* P (or *is a family of* P*-objects*) if $\mathbf{K} \otimes k(p)$ has property P for every point $p \in \operatorname{Spec} R$. Similarly, we say that $\mathbf{K}$ is *geometrically* P if $\mathbf{K} \otimes F$ has P for every geometric point $\operatorname{Spec} F \to \operatorname{Spec} R$.

Although most of the earlier results of this paper were concerned with rank one torsion-free modules, the arguments of this section work for any module in qgr-S_R that satisfies the following hypotheses. As will be seen later, these conditions are satisfied by suitably normalized, stable vector bundles and so the results of this section will also form part of the proof of Theorem 1.6 from the introduction.

Vanishing Condition 5.4. Let R be a commutative ring and pick $S \in \underline{\mathrm{AS}}_3$. Write $(\text{qgr-}S_R)_{\mathrm{VC}}$ for the full subcategory of qgr-S_R consisting of R-flat objects $\mathcal{M}$ satisfying:

$$\mathrm{H}^0(\mathcal{M}(i) \otimes k(p)) = \mathrm{H}^2(\mathcal{M}(i) \otimes k(p)) = 0 \qquad \text{for } i = -1, -2 \quad \text{and} \quad p \in \operatorname{Spec} R.$$

Remark 5.5. Combined with Theorem 4.3(3), this vanishing condition implies that $\mathrm{H}^1(\mathcal{M}(i))$ is a projective R-module for $i = -1, -2$.

The next theorem gives our version of the Beilinson spectral sequence. Later in the section we will extend this result to produce an equivalence of categories between $\mathrm{Monad}(S_R)$ and $(\text{qgr-}S_R)_{\mathrm{VC}}$.

Theorem 5.6. *Assume that $S \in \underline{\mathrm{AS}}_3$ and let R be a commutative noetherian k-algebra. Fix $\mathcal{M} \in (\text{qgr-}S_R)_{\mathrm{VC}}$. Then $\mathcal{M}$ is the cohomology of the monad:*

$$\mathbf{K}(\mathcal{M}): \; 0 \to V_{-1} \otimes_R \mathcal{O}(-1) \to V \otimes_R \mathcal{O} \to V_1 \otimes_R \mathcal{O}(1) \to 0,$$

where $V_{-1} = \mathrm{H}^1(\text{qgr-}S_R, \mathcal{M}(-2))$, $V_1 = \mathrm{H}^1(\text{qgr-}S_R, \mathcal{M}(-1))$ and each V_j is a finitely generated projective R-module.

The module V is defined by (5.11), but the definition is not particularly helpful.

Proof. The first part of the proof follows the argument of [KKO, Theorem 6.6]. By Lemma 5.1, S is a Koszul algebra. In particular, $S = T(S_1)/(\mathcal{R})$ is a quadratic algebra with Koszul dual $S^! = T(S_1^*)/(\mathcal{R}^\perp)$. By [Sm, Theorem 5.9] the *augmented left Koszul resolution* for S is

$$0 \to S(-3) \otimes (S_3^!)^* \to S(-2) \otimes (S_2^!)^* \to S(-1) \otimes (S_1^!)^* \to S \otimes (S_0^!)^* \to k \to 0.$$

Define Ω^1 to be the cohomology of this Koszul complex truncated at $S(-1) \otimes (S_1^!)^*$; equivalently, Ω^1 is defined by the exact sequence

$$0 \longrightarrow \Omega^1 \longrightarrow S(-1) \otimes S_1 \longrightarrow S \longrightarrow k \longrightarrow 0. \tag{5.2}$$

Let $\widetilde{\Omega}^1$ denote the image of Ω^1 in S-qgr. Similarly, given an S-bimodule M, regard M as a left S-module and write $\widetilde{M}$ for the image of M in S-qgr.

Define the *diagonal bigraded algebra of* S to be $\Delta = \bigoplus_{i,j} \Delta_{ij}$, where $\Delta_{ij} = S_{i+j}$. Then, [KKO, Equation 11, p.402] combined with the above observations proves that the following complex of bigraded S-bimodules is exact:

$$0 \to S(-1) \boxtimes S(-2) \to \Omega^1(1) \boxtimes S(-1) \to S \otimes S \to \Delta \to 0. \tag{5.3}$$

Here, $\boxtimes$ stands for external tensor product. To save space we write (5.3) as

$$0 \to K^{-2} \to K^{-1} \to K^0 \to \Delta \to 0.$$

Let $\mathcal{M} \in (\text{qgr-}S_R)_{\text{VC}}$. In the commutative setting, the Beilinson spectral sequence has E_1^{pq} term $\mathrm{H}^q\left(\mathbf{P}^2, \mathcal{M} \otimes \widetilde{\Omega}^{-p}(-p)\right) \otimes_k \mathcal{O}(p)$ and this converges to $\mathcal{M}$. Unfortunately, this does not make sense in our situation: Ω^1 is only a left S-module and so $\mathcal{M} \otimes \widetilde{\Omega}^1(1)$ is no longer a module. We circumvent this problem by using Čech cohomology as formulated in Subsection 4.2. Recall from Remark 4.4 that $\mathrm{H}^i(\text{Qgr-}S_R, \mathcal{M}) = \mathrm{H}^i(\text{Qgr-}S, \mathcal{M})$, which we write as $\mathrm{H}^i(\mathcal{M})$. By Theorem 4.5, $\mathrm{H}^i(\mathcal{M})$ can—and will—be computed as the ith cohomology group $\mathrm{H}^i(\mathcal{M} \otimes_S \mathbf{C}^\bullet)_0$.

Given a module $\mathcal{M} \in (\text{qgr-}S_R)_{\text{VC}}$, write $M = \Gamma^*(\mathcal{M})$ and consider the augmented double complex

$$M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S K^\bullet \longrightarrow M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S \Delta \longrightarrow 0 \tag{5.4}$$

As in [OSS] and [KKO] we have shifted M by (-1) to facilitate the cohomological computations. We also note that, as $- \otimes \mathbf{C}^j$ kills finite dimensional modules, $\mathcal{M}(-1) \otimes_S \mathbf{C}^\bullet$ is well-defined (and equals $M(-1) \otimes_S \mathbf{C}^\bullet$).

Since the sequence $K^\bullet \to \Delta$ is a sequence of graded S-bimodules, each term in (5.4) is also bigraded and so we may take the degree zero summand ${}_0\{-\}$ under the left gradation and then take the image in $\text{Qgr-}S_R$ as right modules. These last two operations are exact functors and we will need to compute the cohomology of the spectral sequence obtained from the resulting double complex in $\text{Qgr-}S_R$:

$$\mathbf{C}^{\bullet,\bullet} = \pi({}_0\{M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S K^\bullet\}). \tag{5.5}$$

We begin by considering the first filtration of this complex. Thus, for fixed i, consider the ith row of (5.4):

$$M(-1) \otimes_S \mathbf{C}^i \otimes_S K^\bullet \to M(-1) \otimes_S \mathbf{C}^i \otimes_S \Delta \to 0. \tag{5.6}$$

As a left S-module, $\Delta \cong \bigoplus_{j \geq 0} S\Delta_{0j} \cong \bigoplus_{j\geq 0} S(j)_{\geq 0}$. Although Δ is not a free left S-module, each summand has finite codimension inside $S(j)$ and so the factor $S(j)/S(j)_{\geq 0}$ is annihilated by tensoring with $\mathbf{C}^i$. Hence, each $\mathbf{C}^i \otimes_S \Delta$ is a flat left S-module. Using (5.2) the same is true of $\mathbf{C}^i \otimes_S \Omega^1$. Therefore (5.6) is also exact. This shows that the spectral sequence associated to (5.5) has E_1 term:

$$E_1^{pq} = \mathrm{H}^q\left(\pi({}_0\{M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S K^p\})\right) \Rightarrow \mathrm{H}^{p+q}\left(\pi({}_0\{M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S \Delta\})\right). \tag{5.7}$$

We emphasize that this is a spectral sequence of objects in $\text{Qgr-}S_R$.

Consider the right hand side of (5.7). For any $r \in \mathbb{Z}$, note that ${}_r\{M(-1) \otimes_S \mathbf{C}^i\} = M(-1)_t \mathbf{C}^i_{r-t}$ for $t \gg 0$ and ${}_{-r}\Delta = \bigoplus_{j \geq 0} \Delta_{-r,j} = \bigoplus_{j\geq 0} S_{j-r}$. Thus, as right S-modules

$${}_0\{M(-1) \otimes_S \mathbf{C}^i \otimes_S \Delta\} \cong \bigoplus_{j \geq 0} \sum_{t,r} M(-1)_t \mathbf{C}^i_{r-t} S_{j-r} = M(-1)\mathbf{C}^i_{\geq 0}.$$

By Theorem 4.5, $\pi({}_0\{M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S \Delta\})$ therefore has cohomology groups

$$\mathrm{H}^i(\pi({}_0\{M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S \Delta\})) = \mathbf{H}^i\left(\mathcal{M}(-1)\right)_{\geq 0} = \bigoplus_{j \geq 0} \mathrm{H}^i\left(\mathcal{M}(j-1)\right). \tag{5.8}$$

Here, $\mathrm{H}^i\left(\text{Qgr-}S_R, \mathcal{M}(j-1)\right) = 0$ for $i > 0$ and $j \gg 0$. Consequently, in $\text{Qgr-}S_R$, the only nonzero cohomology group in (5.8) is

$$\pi\, \mathrm{H}^i(\pi({}_0\{M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S \Delta\})) = \pi(M(-1)) = \mathcal{M}(-1).$$

Thus we have proved:

(5.9) *The spectral sequence* (5.7) *converges to* $\mathcal{M}(-1)$ *concentrated in degree* 0.

We now compute all the terms E_1^{pq} in (5.7). By Condition 5.4, $\mathrm{H}^i\left(\mathcal{M}(-j)\right) = 0$ for $i = 0, 2$ and $j = 1, 2$. Since $K^{-2} = S(-1) \otimes_k S(-2)$ and $K^0 = S \otimes_k S$, this means that $E_1^{pq} = 0$ when both $q \in \{0, 2\}$ and $p \in \{0, -2\}$. By Lemma 2.3(6), the cohomology is also zero for any $q \geq 3$ and by (5.3), it is zero for $p \notin \{0, -1, -2\}$. Thus, it remains to consider the terms E^{pq} when either p or q equals 1.

Sublemma 5.7. *Let $M = \Gamma^*(\mathcal{M})$, as in the proof of Theorem 5.6. Then:*

(1) $\mathrm{H}^i(M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S \Omega^1(1)) = 0$ *for $i = 0, 2$.*
(2) $\mathrm{H}^1(M(-1) \otimes_S \mathbf{C}^\bullet \otimes_S \Omega^1(1))$ *is a finitely generated projective R-module.*

We postpone the proof of the sublemma until we have completed the proof of the theorem. By Sublemma 5.7(1) and the observations beforehand, $E_1^{pq} = 0$ unless $q = 1$. In other words, the spectral sequence (5.7) is simply the complex

$$(5.10) \qquad 0 \to \mathrm{H}^1\big(\mathcal{M}(-2)\big) \otimes_k \mathcal{O}_S(-2) \to V \otimes_k \mathcal{O}_S \to \mathrm{H}^1\big(\mathcal{M}(-1)\big) \otimes_k \mathcal{O}_S \to 0,$$

where

$$(5.11) \qquad V = \mathrm{H}^1\big(\pi({}_0\{M(-1) \otimes \mathbf{C}^\bullet \otimes \Omega^1(1)\})\big) \otimes_k \mathcal{O}_S(-1).$$

By the sublemma, V is a finitely generated projective R-module, while $\mathrm{H}^1(\mathcal{M}(-i))$ is a projective R-module for $i = -1, -2$ by Remark 5.5. Thus (5.9) implies that (5.10) is a monad whose cohomology is precisely $\mathcal{M}(-1)$. Shifting the degree by 1 therefore gives the desired complex $\mathbf{K}(\mathcal{M})$. This completes the proof of Theorem 5.6, modulo the proof of the sublemma. $\square$

Proof of Sublemma 5.7. We first prove the following assertion:

$$(5.12) \quad \text{If } \mathcal{F} \in \text{Gr-}S_R, \text{ then } \mathbf{H}^i(\mathcal{F} \otimes_S \mathbf{C}^\bullet \otimes_S \Omega^1(1)) = \underline{\mathrm{Ext}}^i_{\text{qgr-}S_R}\big((\widetilde{\Omega}^1(1))^\vee, \pi\mathcal{F}\big),$$

where $\mathcal{N}^\vee = \pi\underline{\mathrm{Hom}}_{S\text{-qgr}}(N, \mathcal{O})$ for $\mathcal{N} \in S$-qgr, in the sense of (3.1).

By (5.2), $\widetilde{\Omega}^1$ is a vector bundle and so $\mathbf{C}^\bullet \otimes \Omega^1(1)$ is a complex of flat S-modules. Hence the functors

$$(5.13) \qquad M \mapsto \mathbf{H}^i\big(M \otimes_S \mathbf{C}^\bullet \otimes \Omega^1(1)\big)$$

form a cohomological δ-functor from Qgr-S_R to Gr-R. If $I \in$ Qgr-S_R is injective then Theorem 4.5 implies that $I \otimes_S \mathbf{C}^\bullet$ is exact in cohomological degrees greater than zero. Since $- \otimes_S \widetilde{\Omega}^1(1)$ is an exact functor, it follows that (5.13) is effaceable for all $i > 0$. Thus [Ha, Theorem III.1.3A] implies that the functors (5.13) are the right derived functors of $M \mapsto \mathbf{H}^0\big(M \otimes_S \mathbf{C}^\bullet \otimes \Omega^1(1)\big)$.

Recall the localizations Q_w defined in Subsection 4.2. If $\mathcal{F} \in$ gr-S_R and $N \in S_R$-qgr, then there is a natural map

$$\Psi_{\mathcal{F},N} : \mathcal{F} \otimes Q_w \otimes N \to \underline{\mathrm{Hom}}_{\text{Qgr-}S_R}(N^\vee, \pi\mathcal{F} \otimes Q_w)$$

of graded R-modules given by $f \otimes q \otimes n \mapsto \big[\phi \mapsto f \otimes q \cdot \phi(n)\big]$. The map $\Psi_{\mathcal{F},N}$ is easily seen to be compatible with homomorphisms $N \to N'$ and with the maps $Q_w \to Q'_w$ in the Čech complex. Observe that, if $N = \mathcal{O}_{S_R}(k)$ for some k, then $\Psi_{\mathcal{F},N}$ is an isomorphism: indeed, $N^\vee = \mathcal{O}_{S_R}(-k)$ and so $\Psi_{\mathcal{F},N}$ reduces to the isomorphism $\mathcal{F} \otimes Q_w(k) = \underline{\mathrm{Hom}}_{\text{Qgr-}S_R}(\mathcal{O}_{S_R}(-k), \mathcal{F} \otimes Q_w)$.

Dualizing (5.2) gives an exact sequence $0 \to \mathcal{O} \to \mathcal{O}(1) \otimes S_1^* \to (\widetilde{\Omega}^1)^\vee \to 0$ in S-qgr. After tensoring with R, which we suppress in our notation, this gives the

exact sequence

$$0 \to \underline{\mathrm{Hom}}\big((\widetilde{\Omega}^1)^\vee, \pi\mathcal{F}\otimes Q_w\big) \to \underline{\mathrm{Hom}}\big(\mathcal{O}(1)\otimes S_1^*, \pi\mathcal{F}\otimes Q_w\big) \to \underline{\mathrm{Hom}}\big(\mathcal{O}, \pi\mathcal{F}\otimes Q_w\big), \tag{5.14}$$

where all the Homs are in Qgr-S_R. On the other hand, by (5.2), the sequence

$$0 \to \mathcal{F}\otimes Q_w \otimes \widetilde{\Omega}^1 \to \mathcal{F}\otimes Q_w\otimes \mathcal{O}(-1)\otimes S_1 \to \mathcal{F}\otimes Q_w\otimes\mathcal{O} \tag{5.15}$$

is exact since $\mathcal{O}_{S_R}$ is flat in qgr-S_R. The maps $\Psi_{\mathcal{F},\bullet}$ give a homomorphism of complexes from (5.15) to (5.14) that is an isomorphism in the middle and right-hand columns. Thus $\Psi_{\mathcal{F},\Omega^1}$ is also an isomorphism. Since this is true for every Q_w compatibly with the maps in the Čech complex, Theorem 4.5 implies that $\underline{\mathrm{Hom}}_{\mathrm{Qgr}\text{-}S_R}\big((\widetilde{\Omega}^1)^\vee, \pi\mathcal{F}\big) = \mathbf{H}^0\big(\mathcal{F}\otimes\mathbf{C}^\bullet\otimes\Omega^1\big)$ for every graded S_R-module $\mathcal{F}$. Thus $\mathcal{F}\mapsto \underline{\mathrm{Hom}}_{\mathrm{Qgr}\text{-}S_R}\big((\widetilde{\Omega}^1)^\vee, \pi\mathcal{F}\big)$ and $\mathcal{F}\mapsto \mathbf{H}^0\big(\mathcal{F}\otimes\mathbf{C}^\bullet\otimes\Omega^1\big)$ have the same right-derived functors and (5.12) follows.

We now turn to the proof of the sublemma. The significance of (5.12) is that we can apply Theorem 4.3. Indeed, by parts (1) and (3) of that theorem, the sublemma will be immediate if we can prove that

$$\mathrm{H}^i(M(-1)\otimes k(p)\otimes_S \mathbf{C}^\bullet\otimes_S \Omega^1(1)) = 0 \qquad \text{for } i=0,2 \text{ and } p\in\operatorname{Spec} R. \tag{5.16}$$

In order to prove (5.16), set $M' = M\otimes k(p)$, for some $p\in\operatorname{Spec}R$. As before, $\mathbf{C}^\ell$ kills finite dimensional left S-modules and $\mathbf{C}^\ell\otimes S(j)$ is a flat left S-module. Thus, tensoring the shift of (5.2) with $M'(-1)\otimes_S\mathbf{C}^\bullet$ gives the exact sequence

$$0\to M'(-1)\otimes\mathbf{C}^\bullet\otimes\Omega^1(1)\to M'(-1)\otimes\mathbf{C}^\bullet\otimes S\otimes_k S_1 \to M'(-1)\otimes\mathbf{C}^\bullet\otimes S(1)\to 0. \tag{5.17}$$

Taking homology gives the exact sequence

$$0\to \mathrm{H}^0\left(M'(-1)\otimes_S\mathbf{C}^\bullet\otimes\Omega^1(1)\right)\to \mathrm{H}^0\left(M'(-1)\otimes_S\mathbf{C}^\bullet\otimes_S S\otimes_k S_1\right). \tag{5.18}$$

By Theorem 4.5, the final term of (5.18) equals $\mathrm{H}^0\left(\mathcal{M}(-1)\otimes k(p)\right)\otimes_k S_1$ which, by Condition 5.4, is zero. Thus (5.16) holds for $i=0$.

By construction, $\Omega^1(1)$ can also be included in the exact sequence

$$0\to S(-2)\to S(-1)\otimes(S_2^!)^*\to\Omega^1(1)\to 0. \tag{5.19}$$

Now, $\mathrm{H}^3(\mathrm{Qgr}\text{-}S,-)=0$. Thus, if we tensor (5.19) with $M'(-1)\otimes_S\mathbf{C}^\bullet$ and take homology, we obtain the exact sequence

$$\mathrm{H}^2\left(M'(-1)\otimes_S\mathbf{C}^\bullet\otimes_S S(-1)\otimes_k(S_2^!)^*\right)\to\mathrm{H}^2\left(M'(-1)\otimes_S\mathbf{C}^\bullet\otimes_S\Omega^1(1)\right)\to 0.$$

But now the first term equals $\mathrm{H}^2\left(\mathcal{M}\otimes k(p)(-2)\right)$ which, by Condition 5.4, is zero. Thus (5.16) holds for $i=2$ and so (5.16) is true. This therefore completes the proof of both the sublemma and Theorem 5.6. □

We next show that the functor $\mathcal{M}\mapsto\mathbf{K}(\mathcal{M})$ induces an equivalence of categories.

Theorem 5.8. *Suppose that $S\in\underline{\mathrm{AS}}_3$ and that R is a commutative noetherian k-algebra. Then:*

1. *The functor H^0 induces an equivalence of categories from* Monad(S_R) *to* $(\mathrm{qgr}\text{-}S_R)_{\mathrm{VC}}$*, with inverse $\mathcal{M}\mapsto\mathbf{K}(\mathcal{M})$.*
2. *Let $R\to R'$ be a homomorphism of commutative noetherian k-algebras. Then, as functors from* Monad(S_R) *to* $(\mathrm{qgr}\text{-}S_R)_{\mathrm{VC}}$*, the composite functors $(-\otimes_R R')\circ\mathrm{H}^0$ and $\mathrm{H}^0\circ(-\otimes_R R')$ are naturally equivalent.*

Before proving the theorem, we need several lemmas.

Lemma 5.9. *Let R be a commutative noetherian k-algebra and $S \in \underline{\mathrm{AS}}_3$. If $\mathbf{K}$ is a monad for S_R, then $\mathcal{M} = \mathrm{H}^0(\mathbf{K})$ is R-flat. Moreover, $\mathrm{H}^i(\mathbf{K}) \otimes_R N = \mathrm{H}^i(\mathbf{K} \otimes_R N)$ for every R-module N and every i.*

Proof. Keep the notation of (5.1) and set $\mathcal{B} = \ker(B_{\mathbf{K}})$. From the exact sequence $0 \to \mathcal{B} \to \mathcal{O} \otimes V_0 \to \mathcal{O}(1) \otimes V_1 \to 0$ it is clear that $\mathcal{B}$ is R-flat. It follows from [AZ2, Proposition C1.4] that $\ker(B_{\mathbf{K}}) \otimes N = \ker(B_{\mathbf{K}} \otimes N)$.

Now consider the exact sequence $0 \to \mathcal{O}(-1) \otimes V_{-1} \to \mathcal{B} \to \mathcal{M} \to 0$. By Lemma 2.3(3), this induces the exact sequence of R-modules

$$0 \longrightarrow \mathrm{H}^0(\mathcal{O}(-1+n) \otimes V_{-1}) \xrightarrow{\theta} \mathrm{H}^0(\mathcal{B}(n)) \longrightarrow \mathrm{H}^0(\mathcal{M}(n)) \longrightarrow 0$$

for all $n \gg 0$. In order to prove that $\mathcal{M}$ is R-flat, it suffices to prove that $\mathrm{H}^0(\mathcal{M}(n))$ is R-flat for all n sufficiently large [AZ2, Lemma E5.3]. Thus, by [Ei, Theorem 6.8], it suffices to prove that $\theta \otimes_R k(p)$ is injective for all $n \gg 0$ and all $p \in \operatorname{Spec} R$. Theorem 4.3(3) implies that $\theta \otimes_R k(p)$ is the natural map

$$\mathrm{H}^0\left(\mathcal{O}(-1+n) \otimes V_{-1} \otimes k(p)\right) \to \mathrm{H}^0\left(\mathcal{B}(n) \otimes k(p)\right) = \mathrm{H}^0\left(\ker(B_{\mathbf{K}\otimes k(p)})(n)\right).$$

This map is injective since $A_{\mathbf{K}\otimes k(p)}$ is injective. Thus $\mathcal{M} = \mathrm{H}^0(\mathbf{K})$ is R-flat.

By [AZ2, Proposition C1.4], again, it follows that

$$0 \to \mathcal{O}(-1) \otimes V_{-1} \otimes N \to \ker(B_{\mathbf{K}} \otimes N) \to \mathcal{M} \otimes N \to 0$$

is exact for any R-module N. In particular, $\mathcal{M} \otimes N = \mathrm{H}^0(\mathbf{K} \otimes N)$ and $A_{\mathbf{K}\otimes N}$ is injective. Since $B_{\mathbf{K}}$ is surjective, $B_{\mathbf{K}\otimes N}$ is also surjective. $\square$

Lemma 5.10. *Keep the hypotheses of Theorem 5.8. Let $\mathbf{K}$ be a monad and set $\mathcal{M} = \mathrm{H}^0(\mathbf{K})$. Then for every $p \in \operatorname{Spec} R$, $\mathrm{H}^0(\mathcal{M}(-i) \otimes k(p)) = 0$ for $i \geq 1$ and $\mathrm{H}^2(\mathcal{M}(j) \otimes k(p)) = 0$ for $j \geq -2$.*

Proof. By Lemma 5.9, $\mathcal{M} \otimes k(p) = \mathrm{H}^0(\mathbf{K}) \otimes k(p) = \mathrm{H}^0(\mathbf{K} \otimes k(p))$ for any point p of $\operatorname{Spec} R$. Thus it suffices to prove the claim for a monad $\mathbf{K}$ in qgr-S. Keep the notation of (5.1).

Write $\mathcal{K} = \ker(B_{\mathbf{K}})$, in the notation of (5.1). Then the exact sequence

$$0 \to \mathcal{O}(-1-i) \otimes V_{-1} \to \mathcal{K}(-i) \to \mathcal{M}(-i) \to 0$$

induces an exact sequence $\mathrm{H}^0(\mathcal{K}(-i)) \to \mathrm{H}^0(\mathcal{M}(-i)) \to \mathrm{H}^1(\mathcal{O}(-1-i))$. Since $\mathcal{K}(-i) \subseteq V_0 \otimes \mathcal{O}(-i)$, one has $\mathrm{H}^0(\mathcal{K}(-i)) = 0$ for all $i > 0$. On the other hand, $\mathrm{H}^1(\mathcal{O}(-1-i)) = 0$ for all i, by Lemma 2.3(5). Thus, $\mathrm{H}^0(\mathcal{M}(-i)) = 0$ for $i \geq 1$. By Lemma 2.3(5), $\mathrm{H}^2(\mathcal{O}(j) \otimes V_0) = 0$ for $j \geq -2$ while $\mathrm{H}^3(\mathcal{N}) = 0$ for any $\mathcal{N} \in$ qgr-S. Thus the long exact sequence in homology associated to the surjection $\mathcal{O}(j) \otimes V_0 \twoheadrightarrow \operatorname{coker}(A_{\mathbf{K}})(j)$ shows that $\mathrm{H}^2(\operatorname{coker}(A_{\mathbf{K}})(j)) = 0$ for $j \geq -2$.

Finally, the exact sequence

$$0 \to \mathcal{M}(j) \to \operatorname{coker}(A_{\mathbf{K}})(j) \to \mathcal{O}(1+j) \otimes V_1 \to 0$$

induces the exact sequence

$$\mathrm{H}^1(\mathcal{O}(1+j) \otimes V_1) \to \mathrm{H}^2(\mathcal{M}(j)) \to \mathrm{H}^2(\operatorname{coker}(A_{\mathbf{K}})(j)).$$

By Lemma 2.3(5), respectively the conclusion of the last paragraph, the outside terms vanish for $j \geq -2$. Thus, the middle term is also zero. $\square$

The last two lemmas imply that H^0 maps $\mathrm{Monad}(S_R)$ to $(\text{qgr-}S_R)_{\mathrm{VC}}$. We next want to prove that H^0 is a full and faithful functor on monads, which will complete the proof of Theorem 5.8(1). This follows from the following more general fact.

Lemma 5.11. *Keep the hypotheses of Theorem 5.8 and suppose that* $\mathbf{K}$ *is a monad and* $\mathbf{L}$ *is a complex of the form*

$$\mathbf{L}: \quad W_{-1}\otimes\mathcal{O}(-1)\to W_0\otimes\mathcal{O}\to W_1\otimes\mathcal{O}(1)$$

with each W_i *a finitely generated* R*-module and* $\mathrm{H}^n(\mathbf{L})=0$ *for* $n\neq 0$*. Let* $\mathcal{E}=\mathrm{H}^0(\mathbf{K})$ *and* $\mathcal{F}=\mathrm{H}^0(\mathbf{L})$*. Then for any* i*,* $\mathrm{Ext}^i(\mathcal{E},\mathcal{F})=\mathrm{H}^i(\mathrm{Hom}^\bullet(\mathbf{K},\mathbf{L}))$*.*

Proof. Let L^n denote the nth term in $\mathbf{L}$ and choose an injective resolution $L^n\to I^n_\bullet$ for each n; these combine to give a double complex $(I^\bullet_\bullet)$ the total complex of which is an injective resolution of $\mathbf{L}$. Let d_1 and d_2 denote the two differentials (the first "in the $\mathbf{L}$-direction" and the second "in the I-direction"), with signs adjusted as usual so that $d_1d_2+d_2d_1=0$. Let d_1 denote the given differential on $\mathbf{K}$ and define a second differential d_2 on $\mathbf{K}$ to be zero everywhere.

Define a double complex by setting $\mathcal{C}^{p,q}=\bigoplus_n \mathrm{Hom}(K^n,I^{n+p}_q)$ and defining differentials by $\delta_i(f)=d_i\circ f-(-1)^{\deg(f)}f\circ d_i$. The associated total complex then satisfies $\mathrm{Tot}(\mathcal{C}^{\bullet,\bullet})=\mathrm{Hom}^\bullet(\mathbf{K},\mathrm{Tot}(I^\bullet_\bullet))$, which yields $\mathrm{H}^i(\mathrm{Tot}(\mathcal{C}^{\bullet,\bullet}))=\mathrm{Ext}^i(\mathcal{E},\mathcal{F})$ since $\mathbf{K}$ is quasi-isomorphic to $\mathcal{E}$. On the other hand, the double complex has associated spectral sequence with $E_2^{\bullet,\bullet}=H_I(H_{II}(\mathcal{C}^{\bullet,\bullet}))$. Now

$$H_{II}^{p,q}(\mathcal{C}^{\bullet,\bullet})=\bigoplus_n \mathrm{Ext}^q(K^n,L^{n+p})=\bigoplus_{n=-1}^{1}\mathrm{H}^q(\mathcal{O}(p))\otimes V_n^*\otimes W_{n+p},$$

which vanishes by hypothesis when $q\neq 0$ and satisfies $H_{II}^{p,0}=\bigoplus_n\mathrm{Hom}(K^n,L^{n+p})$. The spectral sequence thus degenerates at E_2, completing the proof. $\square$

Proof of Theorem 5.8. Lemmas 5.9 and 5.10 show that H^0 maps $\mathrm{Monad}(S_R)$ to $(\mathrm{qgr}\text{-}S_R)_{\mathrm{VC}}$, while Theorem 5.6 shows that H^0 is surjective on objects. Lemma 5.11 implies that H^0 is fully faithful, which by [Pp, Theorem I.5.3] is sufficient to prove part (1). Part (2) of the theorem is immediate from Lemma 5.9. $\square$

6. Semistable Modules and Kronecker Complexes

We want to use the results of the previous section to construct a projective coarse moduli space for modules over AS regular rings. As will be seen in Lemma 6.4, the construction of Theorem 5.8 applies to a large class of flat families of modules including all torsion-free modules of rank one, up to a shift. To construct the moduli space as a projective scheme, it will then suffice to give a convenient realization of a parameter space for monads. The condition that the pair of maps A_K and B_K in (5.1) actually define a monad rather than just a complex is awkward to describe directly in terms of linear algebra. The standard way round this difficulty [DL] is to characterize monads as so-called Kronecker complexes satisfying an appropriate stability condition. This approach also works in our noncommutative setting and the details are given in this section.

6.1. **Modules in** qgr-S**.** There are several invariants that can be attached to a module $\mathcal{M}\in\mathrm{qgr}\text{-}S$ for $S\in\underline{\mathrm{AS}}_3$ and we begin by describing them.

As before, we write $\mathcal{O}_S=\pi(S)\in\mathrm{qgr}\text{-}S$. The *Euler characteristic* of $\mathcal{M}$ is defined to be $\chi(\mathcal{M})=\sum(-1)^ih^i(\mathcal{M})$, where $h^i(\mathcal{M})=\dim\mathrm{H}^i(\mathcal{M})$, and the *Hilbert polynomial* of $\mathcal{M}$ is $p_{\mathcal{M}}(t)=\chi(\mathcal{M}(t))$. If $\mathcal{M}$ is nonzero and torsion-free, the *normalized Hilbert polynomial* of $\mathcal{M}$ is $p_{\mathcal{M}}(t)/\operatorname{rk}(\mathcal{M})$. The other invariant that

will be frequently used is the first Chern class, as defined on page 16. When qgr-$S \simeq \operatorname{coh} \mathbf{P}^2$, these definitions coincide with the usual commutative ones.

The next lemma gives some standard properties of these invariants.

Lemma 6.1. (1) *If $\mathcal{M} \in$ qgr-S then $p_{\mathcal{M}}$ is a polynomial. The Hilbert polynomial is additive in exact sequences and has positive leading coefficient.*

(2) *If $\mathcal{M} \in$ qgr-S then*

$$p_{\mathcal{M}}(t) = \frac{1}{2}\operatorname{rk}(\mathcal{M})t(t+1) + \big(c_1(\mathcal{M}) + \operatorname{rk}(\mathcal{M})\big)t + \chi(\mathcal{M}). \tag{6.1}$$

Proof. To begin with, assume that $\mathcal{M} = \mathcal{O}_S(m)$ for some $m \in \mathbb{Z}$. Since $S \in \underline{\mathrm{AS}}_3$, S has the Hilbert series of a polynomial ring in 3 variables and so $\mathrm{H}^0(\text{qgr-}S, \mathcal{O}_S(m)) = \mathrm{H}^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(m))$ for all m. By Lemma 2.3(5) and Serre duality (Proposition 2.4), $\mathrm{H}^i(\text{qgr-}S, \mathcal{O}_S(m)) = \mathrm{H}^i(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(m))$, for $i \geq 1$ and $m \in \mathbb{Z}$. Thus, by Riemann-Roch, as stated in [LP2, p.154], the lemma holds for $\mathcal{M} = \mathcal{O}(m)$.

Now let $\mathcal{M}$ be arbitrary. As usual, additivity of the Hilbert polynomial on exact sequences follows from the long exact cohomology sequence. Thus, both sides of (6.1) are additive on exact sequences. Since S has finite global dimension, every finitely generated graded S-module admits a finite free resolution. By additivity it therefore suffices to prove the lemma for $\mathcal{M} = \mathcal{O}_S(m)$, as we have done. □

Corollary 6.2. *Let $\chi(\mathcal{E}, \mathcal{F}) = \sum_i (-1)^i \dim \operatorname{Ext}^i(\mathcal{E}, \mathcal{F})$ for $\mathcal{E}, \mathcal{F} \in$ qgr-S. Then*

$$\chi(\mathcal{E}, \mathcal{F}) = \operatorname{rk}(\mathcal{E})[\chi(\mathcal{F}) - \operatorname{rk}(\mathcal{F})] - c_1(\mathcal{E})[3\operatorname{rk}(\mathcal{F}) + c_1(\mathcal{F})] + \chi(\mathcal{E})\operatorname{rk}(\mathcal{F}). \tag{6.2}$$

Proof. The formula reduces to (6.1) when $\mathcal{E} = \mathcal{O}(k)$ and $\mathcal{F} = \mathcal{O}(\ell)$. Since both sides of (6.2) are separately additive in $\mathcal{E}$ and $\mathcal{F}$, the formula follows by taking resolutions of both $\mathcal{E}$ and $\mathcal{F}$ by direct sums of $\mathcal{O}(n)$'s. □

A torsion-free module $\mathcal{M} \in$ qgr-S is *normalized* if $-\operatorname{rk}(\mathcal{M}) < c_1(\mathcal{M}) \leq 0$. If $\mathcal{M}$ is torsion-free then Lemma 3.7 implies that $c_1(\mathcal{M}(1)) = c_1(\mathcal{M}) + \operatorname{rk}(\mathcal{M})$ and so there is a unique normalized shift of $\mathcal{M}$. In particular, a torsion-free, rank 1 module $\mathcal{M}$ is normalized if and only if $c_1(\mathcal{M}) = 0$.

Order polynomials lexicographically and write *(semi)stable* to mean "stable, respectively semistable". A torsion-free module $\mathcal{M} \in$ qgr-S is *(semi)stable*, if, for every proper submodule $0 \neq \mathcal{F} \subset \mathcal{M}$, one has $\operatorname{rk}(\mathcal{M})p_{\mathcal{F}} - \operatorname{rk}(\mathcal{F})p_{\mathcal{M}} < 0$ (respectively ≤ 0). $\mathcal{M}$ is called *geometrically (semi)stable* if $\mathcal{M} \otimes \overline{k}$ is (semi)stable where $\overline{k}$ is an algebraic closure of k. $\mathcal{M}$ is called *μ-(semi)stable* if $\operatorname{rk}(\mathcal{M})c_1(\mathcal{F}) - \operatorname{rk}(\mathcal{F})c_1(\mathcal{M}) < 0$ (respectively ≤ 0).

If $\mathcal{F} \subset \mathcal{E}$ is de-(semi)stabilizing, then $\mathcal{F} \otimes \overline{k} \subset \mathcal{E} \otimes \overline{k}$ is de-(semi)stabilizing and so a geometrically (semi)stable module is (semi)stable. One also has the standard implications

$$\mu\text{-stable} \;\Rightarrow\; \text{stable} \;\Rightarrow\; \text{semistable} \;\Rightarrow\; \mu\text{-semistable}.$$

As Lemma 6.4 will show, semistable modules have tightly controlled cohomology. The argument uses the following standard consequence of [Ru, Theorem 3].

Lemma 6.3. *Let $\mathcal{M} \in$ qgr-S be torsion-free and semistable. Then $\mathcal{M}$ admits a* Jordan-Holder filtration *in the sense that there exists a filtration*

$$\{0\} = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \cdots \subsetneq \mathcal{M}_k = \mathcal{M}$$

such that

(1) *each* $\operatorname{gr}_i(\mathcal{M}) = \mathcal{M}_i/\mathcal{M}_{i-1}$ *is a torsion-free stable module in* qgr-S*, and*
(2) $p_{\mathcal{M}}/\operatorname{rk}(\mathcal{M}) = p_{\operatorname{gr}_i(\mathcal{M})}/\operatorname{rk}(\operatorname{gr}_i(\mathcal{M}))$ *for every* i. □

Lemma 6.4. *Suppose that* $\mathcal{M} \in$ qgr-S *is torsion-free. Then:*
(1) *If* $\mathcal{M}$ *is* μ*-semistable and normalized, then* $\mathrm{H}^0(\mathcal{M}(-i)) = 0$ *for* $i \geq 1$ *and* $\mathrm{H}^2(\mathcal{M}(i)) = 0$ *for* $i \geq -2$.
(2) *If* $\mathcal{M}$ *is semistable and normalized, then either* $\mathcal{M} \cong \mathcal{O}^r$ *for* $r = \operatorname{rk}(\mathcal{M})$ *or* $\mathrm{H}^0(\mathcal{M}) = 0$.

Proof. Although the proof is very similar to that of [DL, Lemme 2.1] we will give it since it is fundamental to our approach.

(1) If $\mathrm{H}^0(\mathcal{M}(-i)) \neq 0$ for some $i > 0$, then $\mathcal{O}_S(i) \hookrightarrow \mathcal{M}$. As $c_1(\mathcal{M}) \leq 0$, μ-semistability forces

$$\operatorname{rk}(\mathcal{M})c_1(\mathcal{O}(i)) \;\leq\; \operatorname{rk}(\mathcal{M})c_1(\mathcal{O}(i)) - i \cdot c_1(\mathcal{M}) \leq 0,$$

a contradiction. If $\mathrm{H}^2(\mathcal{M}(-3+i)) \neq 0$ for some $i \geq 1$ then Serre duality (Proposition 2.4) implies that there exists $0 \neq \theta \in \operatorname{Hom}(\mathcal{M}, \mathcal{O}(-i))$. It is then a simple exercise to see that $\mathcal{F} = \ker(\theta)$ contradicts the μ-semistability of $\mathcal{M}$.

(2) As $\mathcal{M}$ is μ-semistable, part (1) implies that $p_{\mathcal{M}}(-1) = \chi(\mathcal{M}(-1)) \leq 0$. By (6.1), this forces $\chi(\mathcal{M}) \leq c_1(\mathcal{M}) + \operatorname{rk}(\mathcal{M})$. Now assume that $h^0(\mathcal{M}) \neq 0$; thus $\mathcal{O}_S \hookrightarrow \mathcal{M}$. By μ-semistability and the fact that $c_1(\mathcal{O}_S) = 0$ we obtain $c_1(\mathcal{M}) = 0$. Hence $\chi(\mathcal{M}) \leq \operatorname{rk}(\mathcal{M})$. Conversely, semistability implies that $\operatorname{rk}(\mathcal{M})p_{\mathcal{O}_S} - p_{\mathcal{M}} \leq 0$. Substituting this into (6.1) shows that $\chi(\mathcal{M}) \geq \operatorname{rk}(\mathcal{M})$. Thus, $\chi(\mathcal{M}) = \operatorname{rk}(\mathcal{M})$ and so $\mathcal{M}$ has normalized Hilbert polynomial equal to that of $\mathcal{O}_S$.

By Lemma 6.3, choose a Jordan-Hölder filtration $\{\mathcal{M}_i\}$ of $\mathcal{M}$. By the last paragraph, each $\operatorname{gr}_i \mathcal{M}$ is stable and has normalized Hilbert polynomial equal to that of $\mathcal{O}$. Also, $\chi(\operatorname{gr}_i \mathcal{M}) > 0$ by Lemma 6.3(2) but $\mathrm{H}^2(\operatorname{gr}_i \mathcal{M}) = 0$ by part (1) of this proof. Thus, $\operatorname{gr}_i \mathcal{M}$ has a nonzero global section. The corresponding inclusion $\mathcal{O} \hookrightarrow \operatorname{gr}_i \mathcal{M}$ contradicts the stability of $\operatorname{gr}_i \mathcal{M}$ unless $\operatorname{gr}_i \mathcal{M} \cong \mathcal{O}$ for each i. Finally, Lemma 2.3(5) implies that $\operatorname{Ext}^1_{\text{qgr-}S}(\mathcal{O}, \mathcal{O}) = 0$ and hence that $\mathcal{M} \cong \mathcal{O}^r$. □

Remark 6.5. The proof of the lemma shows that, if $\mathcal{M}$ is torsion-free, semistable and normalized then $\chi(\mathcal{M}) \leq c_1(\mathcal{M}) + \operatorname{rk}(\mathcal{M})$.

Combined with the Beilinson spectral sequence, the observations of this section have strong consequences for the cohomology of rank one torsion-free modules.

Corollary 6.6. *Let* $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$*. Assume that* $\mathcal{M} \in$ qgr-S *is a torsion-free, rank one module with* $c_1(\mathcal{M}) = 0$*. Then:*
(1) $\mathcal{M}$ *is stable and* $\mathcal{M} \in (\text{qgr-}S)_{\mathrm{VC}}$*, as defined in Condition 5.4.*
(2) $\dim_k \mathrm{H}^1(\mathcal{M}(-1)) = \dim_k \mathrm{H}^1(\mathcal{M}(-2)) = n$*, where* $n = 1 - \chi(\mathcal{M})$.
(3) *Suppose that* $\mathcal{M} \in \mathcal{V}_S$*; that is, suppose that* $\mathcal{M}|_E$ *is a vector bundle. Then* $\mathcal{M}|_E \cong \big(\mathcal{L} \otimes (\overline{\mathcal{L}})^{-1}\big)^{\otimes n}$ *where* $\overline{\mathcal{L}} = \mathcal{L}^{\sigma^{-1}}$ *and* $n = 1 - \chi(\mathcal{M})$.

Proof. (1) If $0 \neq \mathcal{F} \subsetneq \mathcal{M}$ then $\operatorname{rk}(\mathcal{F}) = \operatorname{rk}(\mathcal{M})$ but $p_{\mathcal{F}} < p_{\mathcal{M}}$, simply because $p(\mathcal{M}/\mathcal{F}) > 0$. Thus, $\mathcal{M}$ is stable. By Lemma 6.4, $\mathcal{M} \in (\text{qgr-}S)_{\mathrm{VC}}$.

(2) By part (1) and Theorem 5.6, $\mathcal{M}$ is the cohomology of a monad, which we assume has the form (5.1). By the additivity of c_1 on exact sequences,

$$0 = c_1(\mathcal{M}) = c_1(\mathcal{O}(-1) \otimes V_{-1}) + c_1(\mathcal{O}(1) \otimes V_1) = \dim V_1 - \dim V_{-1}.$$

Equivalently, $\dim_k \mathrm{H}^1(\mathcal{M}(-1)) = \dim_k \mathrm{H}^1(\mathcal{M}(-2))$. Lemma 2.3(5) and the additivity of χ on exact sequences then imply that $\chi(\mathcal{M}) = 1 - \dim V_1$.

(3) By Lemma 2.5, the restriction of (5.1) to E is a complex of the form

$$0 \to V_{-1} \otimes \mathcal{L}^* \to V_0 \otimes \mathcal{O}_E \to V_1 \otimes \overline{\mathcal{L}} \to 0. \tag{6.3}$$

Equation 5.1 can be split into two short exact sequences of torsion-free modules and by Lemma 2.6 the restrictions of those exact sequences to E are again exact. Thus (6.3) is a complex whose only cohomology is the vector bundle $\mathcal{M}|_E$ in degree zero. Part (3) therefore follows by taking determinants of (6.3). □

6.2. **Kronecker Complexes.** We continue to assume that $S \in \underline{\mathrm{AS}}_3$ in this subsection. To treat moduli of monads, it is more convenient to work with (a priori) more general complexes that satisfy suitable stability properties. In this subsection we define the relevant complexes, called Kronecker complexes, and show that semistability forces them to be monads. Our treatment closely follows [DL].

Definition 6.7. A *Kronecker complex* in qgr-S is a complex of the form

$$\mathbf{K}: \quad \mathcal{O}(-1) \otimes V_{-1} \xrightarrow{A} \mathcal{O} \otimes V_0 \xrightarrow{B} \mathcal{O}(1) \otimes V_1 \tag{6.4}$$

where V_{-1}, V_0 and V_1 are finite-dimensional vector spaces. Clearly a monad is a special case of a Kronecker complex. We index the complex so that $\mathcal{O}(i)$ occurs in cohomological degree i; thus $\mathrm{H}^1(\mathbf{K})$ denotes the homology at $\mathcal{O}(1) \otimes V_1$.

Morphisms of Kronecker complexes are just morphisms of complexes, and so are defined by maps of the defining vector spaces V_i. It is then easy to see that the category of Kronecker complexes in qgr-S is an abelian category.

The invariants we just defined for modules have their natural counterparts for Kronecker complexes. Thus, if $\mathbf{K}$ is a Kronecker complex as in (6.4), the *rank* of $\mathbf{K}$ is defined to be $\mathrm{rk}(\mathbf{K}) = \mathrm{rk}(V_0) - \mathrm{rk}(V_{-1}) - \mathrm{rk}(V_1)$. This can be negative. The *first Chern class* $c_1(\mathbf{K})$ is $c_1(\mathbf{K}) = \mathrm{rk}(V_{-1}) - \mathrm{rk}(V_1)$ and the *Euler characteristic* $\chi(\mathbf{K})$ is $\chi(\mathbf{K}) = \mathrm{rk}(V_0) - 3\,\mathrm{rk}(V_1)$. The *Hilbert polynomial* $p_{\mathbf{K}}$ of $\mathbf{K}$ is given by the formula

$$p_{\mathbf{K}} = \mathrm{rk}(V_0) \cdot p_{\mathcal{O}} - \mathrm{rk}(V_{-1}) \cdot p_{\mathcal{O}(-1)} - \mathrm{rk}(V_1) \cdot p_{\mathcal{O}(1)}.$$

The *normalized Hilbert polynomial* of a Kronecker complex $\mathbf{K}$ of positive rank is $p_{\mathbf{K}}/\,\mathrm{rk}(\mathbf{K})$. If $\mathbf{K}$ is a monad, all these invariants coincide with the corresponding invariants of the cohomology $\mathrm{H}^0(\mathbf{K})$.

The correct notion of (semi)stability for Kronecker complexes is the following.

Definition 6.8. A Kronecker complex $\mathbf{K}$ is *(semi)stable* if, for every proper subcomplex $\mathbf{K}'$ of $\mathbf{K}$, one has $\mathrm{rk}(\mathbf{K})p_{\mathbf{K}'} - \mathrm{rk}(\mathbf{K}')p_{\mathbf{K}} < 0$ (respectively ≤ 0) under the lexicographic order on polynomials. Equivalently:

(1) $\mathrm{rk}(\mathbf{K})c_1(\mathbf{K}') - \mathrm{rk}(\mathbf{K}')c_1(\mathbf{K}) \leq 0$, and
(2) if equality holds in (1) then $\mathrm{rk}(\mathbf{K})\chi(\mathbf{K}') - \mathrm{rk}(\mathbf{K}')\chi(\mathbf{K}) < 0$ (respectively ≤ 0).

A Kronecker complex $\mathbf{K}$ is *geometrically (semi)stable* if $\mathbf{K} \otimes \overline{k}$ is (semi)stable where $\overline{k}$ is an algebraic closure of k. A Kronecker complex $\mathbf{K}$ of rank $r > 0$ is *normalized* if $-r < c_1(\mathbf{K}) \leq 0$.

The following proposition is the main result of this section. It obviously allows us to replace monads by semistable Kronecker complexes in describing S-modules and this will be important since it is much easier to describe the moduli spaces of Kronecker complexes than those of monads.

Proposition 6.9. *Let $S \in \underline{\mathrm{AS}}_3$. Suppose that $\mathbf{K}$ is a semistable normalized Kronecker complex. Then $\mathbf{K}$ is a torsion-free monad in the sense of Definition 5.2.*

Before proving the proposition, we need several preliminaries. Since we are proving a result about qgr-S, it suffices to prove the result when $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$ and we assume this throughout the proof.

The first three results closely follow the strategy from [DL] where similar results are proved in the commutative case.

Lemma 6.10. *Let $S \in \underline{\mathrm{AS}}'_3$ and suppose that $\mathbf{K}$ is a Kronecker complex of the form* (6.4).

(a) *If $B|_E$ is not surjective then $\mathbf{K}$ has a quotient complex of one of the forms:*

$$\begin{array}{llllll} (3) & \mathcal{O}(-1) & \to \mathcal{O}^2 & \to \mathcal{O}(1) & & \textit{which is exact at } \mathcal{O}^2 \\ (4) & 0 & \to \mathcal{O} & \to \mathcal{O}(1) & & \\ (5) & 0 & \to 0 & \to \mathcal{O}(1) & & \\ (6) & 0 & \to \mathcal{O}^2 & \to \mathcal{O}(1) & & \end{array}$$

(b) *Suppose either that $\mathcal{O}(-1)|_E \otimes V_{-1} \xrightarrow{A|_E} \mathcal{O}_E \otimes V_0$ is not injective or that $\operatorname{coker}(A|_E)$ has a simple subobject. Then $\mathbf{K}$ has a subcomplex either of type (3) or of one of the forms*

$$\begin{array}{llll} (1) & \mathcal{O}(-1) & \to 0 & \to 0 \\ (2) & \mathcal{O}(-1) & \to \mathcal{O} & \to 0 \\ (7) & \mathcal{O}(-1) & \to \mathcal{O}^2 & \to 0 \end{array}$$

Remark 6.11. Our numbering of these complexes is chosen to be consistent with that in [DL]. We will refer to a complex of one of the forms above as a *standard complex of type (3)*, etc. Note that the exactness hypothesis of a complex of type (3) implies that the first map in that complex is automatically an injection.

Proof. (a) Assume that $B|_E$ is not surjective. Then there exists a projection $\pi : \mathcal{O}(1) \otimes V_1 \to \mathcal{O}(1)$ such that $(\pi \circ B)|_E$ is not surjective. Since $\mathcal{O}(1)$ is globally generated and $h^0(\mathcal{O}(1)) = 3$, the image $W = (\pi \circ B)(\mathrm{H}^0(\mathcal{O} \otimes V_0))$ of $\mathrm{H}^0(\mathcal{O} \otimes V_0)$ satisfies $\dim W \leq 2$. This induces the commutative diagram of complexes

$$\begin{tikzcd} \mathcal{O} \otimes V_0 \arrow[r, "B"] \arrow[d, "\rho_1"] & \mathcal{O}(1) \otimes V_1 \arrow[r, "\tau"] \arrow[d, "\rho"] & P_p(1) \arrow[d, "="] \\ \mathcal{O} \otimes W \arrow[r, "\overline{B}"] & \mathcal{O}(1) \arrow[r, "\tau'"] & P_p(1) \end{tikzcd} \tag{6.5}$$

where P_p is the module in qgr-$S/Sg \subset$ qgr-S corresponding to some closed point $p \in E$. Moreover, the maps τ and τ' are surjections.

If $\dim W$ is 0 or 1 we obtain a complex of type (5), respectively (4).

So, assume that $\dim W = 2$; thus $\overline{B}$ induces an injection of global sections. We now want to use a result from [ATV2] which requires an algebraically closed field. Thus, let k have algebraic closure F and use "superscript F" to denote $- \otimes_k F$. Let P_q^F denote a simple factor module of P_p^F in qgr-$S^F/S^F g \subset$ qgr-S^F. (Although we do not need it, this module does correspond to a closed point $q \in E^F$.) So the

final row of the commutative diagram gives a complex

$$\mathcal{O}\otimes W^F \xrightarrow{\overline{B}^F} \mathcal{O}^F(1) \xrightarrow{\tau''} P_q^F(1)\to 0, \tag{6.6}$$

where $\overline{B}^F$ still gives an injection of global sections and τ'' is surjective. By [ATV2, Proposition 6.7(iii)] the minimal resolution of $P_q^F(1)$ has the form

$$0\to \mathcal{O}^F(-\epsilon-1)\xrightarrow{\alpha}\mathcal{O}^F\oplus\mathcal{O}^F(-\epsilon)\xrightarrow{\beta}\mathcal{O}^F(1)\xrightarrow{\tau''}P_q^F(1)\to 0, \tag{6.7}$$

for some $\epsilon\geq 0$. Since $\overline{B}^F$ is injective on global sections, $\epsilon=0$. We may therefore assume that $\beta=\overline{B}^F$, in which case (6.7) is an extension of (6.6). Since τ'' factors through $(\tau')^F$, this forces $P_p^F=P_q^F$ to be simple and (6.6) to be exact.

Returning to k-algebras, this implies that the second row of (6.5) is also exact. Applying Γ^* to that sequence gives the complex

$$S\otimes_k W\xrightarrow{\Gamma^*(\overline{B})} S(1)\xrightarrow{\Gamma^*(\tau')}\Gamma(P_p(1)). \tag{6.8}$$

By Lemma 2.3(1), $\Gamma^*(\tau')$ is surjective in high degrees and it cannot be zero in any degree ≥ 1. Since S has Hilbert series $(1-t)^{-3}$, computing the dimension of (6.8) shows that $\Gamma^*(\overline{B})$ is not surjective in degree one. Equivalently, the map α of (6.7) is defined over k.

Summing up, this implies that (6.5) can be extended to the following commutative diagram for which the second row is exact and the columns are surjections:

$$\begin{tikzcd} \mathcal{O}(-1)\otimes V_{-1} \arrow[r, "A"] & \mathcal{O}\otimes V_0 \arrow[r, "B"] \arrow[d, "\rho_1"] & \mathcal{O}(1)\otimes V_1 \arrow[d, "\rho"] \\ \mathcal{O}(-1) \arrow[r, "\overline{A}"] & \mathcal{O}^2 \arrow[r, "\overline{B}"] & \mathcal{O}(1). \end{tikzcd} \tag{6.9}$$

We may extend the vertical maps to a map of Kronecker complexes, thereby showing that $\mathbf{K}$ has a factor of type (3) or type (6).

(b) This follows from part (a) applied to the dual complex. In more detail, first suppose that $\mathbf{K}$ is a Kronecker complex for which $A_{\mathbf{K}}|_E$ is not injective. Then $\mathbf{K}^*$ is a Kronecker complex (of left S-modules!) in which $B_{\mathbf{K}^*}|_E=(A_{\mathbf{K}}|_E)^*$ is not surjective. Thus, we may apply part (a) to obtain a quotient complex of $\mathbf{K}^*$ of one of the given types; dualizing this gives the appropriate subcomplex of $\mathbf{K}$.

On the other hand, suppose that $A_{\mathbf{K}}|_E$ is injective but its cokernel C has a subobject T of finite length. As E is a Gorenstein curve, $\underline{\mathrm{Ext}}^2_E(C/T,\mathcal{O}_E)=0$ but $\underline{\mathrm{Ext}}^1_E(T,\mathcal{O}_E)\neq 0$. Thus, by the long exact sequence in cohomology, $\underline{\mathrm{Ext}}^1_E(C,\mathcal{O}_E)\neq 0$. Consider the exact sequence $0\to P\to Q\to C\to 0$, where $P=\mathcal{O}_E(-1)\otimes V_{-1}$ and $Q=\mathcal{O}_E\otimes V_0$. Dualizing gives the exact sequence

$$0\longrightarrow C^*\longrightarrow Q^*\xrightarrow{\theta}P^*\longrightarrow \underline{\mathrm{Ext}}^1_E(C,\mathcal{O}_E)\longrightarrow 0.$$

Thus, $A_{\mathbf{K}^*}|_E=\theta$ is not surjective. As in the last paragraph, we can now apply part (a) to the dual complex $\mathbf{K}^*$ to find the appropriate subcomplex of $\mathbf{K}$. $\square$

Remark 6.12. One has the following table of numerics for the various standard complexes. Let $r=\mathrm{rank}$, c_1, χ denote the invariants of the complex $\mathbf{K}$ and use

the same letters with primes attached to denote the corresponding invariants of the standard complex $\mathbf{K}'$.

type	r'	c_1'	χ'	$rc_1' - r'c_1$	$r\chi' - r'\chi$	$rp_{\mathbf{K}'} - r'p_{\mathbf{K}}$
(3)	0	0	-1	0	$-r$	$-r$
(4)	0	-1	-2	$-r$	$-2r$	$-r(t+2)$
(5)	-1	-1	-3	$-r+c_1$	$-3r+\chi$	$(-r+c_1)t+(-3r+\chi)$
(6)	1	-1	-1	$-r-c_1$	$-r-\chi$	$(-r-c_1)t+(-r-\chi)$
(1)	-1	1	0	$r+c_1$	χ	$(r+c_1)t+\chi$
(2)	0	1	1	r	r	$rt+r$
(7)	1	1	2	$r-c_1$	$2r-\chi$	$(r-c_1)t+(2r-\chi)$

An immediate consequence of this table is:

Corollary 6.13. *Let $S \in \underline{\mathrm{AS}}_3'$. A semistable, normalized Kronecker complex $\mathbf{K}$ has no quotient complex of types (3–6) and no subcomplex of types (1), (2) or (7).* $\square$

Lemma 6.14. *Let $S \in \underline{\mathrm{AS}}_3'$ and suppose that $\mathbf{K}$ is a normalized semistable Kronecker complex of the form (6.4). Then $\mathbf{K}$ admits a filtration by subcomplexes*

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell \subseteq \mathbf{K}$$

such that

(i) *each F_i/F_{i-1} is of type (3),*
(ii) *each $\mathbf{K}/F_i$ contains no subcomplex of type (1), (2) or (7), and*
(iii) *the first map $A_{\mathbf{K}/F_\ell}|_E$ in $(\mathbf{K}/F_\ell)|_E$ is injective and $\operatorname{coker}(A_{\mathbf{K}/F_\ell}|_E)$ has no simple subobjects in $\operatorname{coh} E$.*

Proof. If $A_{\mathbf{K}}|_E$ is injective and $\operatorname{coker}(A_{\mathbf{K}}|_E)$ is torsion-free then, by Corollary 6.13, the lemma holds with $\ell = 0$. If $A_{\mathbf{K}}|_E$ does not have both these properties, then Lemma 6.10 and Corollary 6.13 imply that $\mathbf{K}$ has a subcomplex F_1 of type (3).

Suppose that we have constructed $0 = F_0 \subset \cdots \subset F_k$ so that each subquotient F_i/F_{i-1} is of type (3). If $A_{\mathbf{K}/F_k}$ satisfies condition (iii) above then we are finished, so suppose not. Then Lemma 6.10 implies that $\mathbf{K}/F_k$ has a subcomplex $\mathbf{K}'/F_k$ of type (1), (2), (3) or (7). Additivity of rank and c_1 implies that

$$\operatorname{rk}(\mathbf{K})c_1(\mathbf{K}') - \operatorname{rk}(\mathbf{K}')c_1(\mathbf{K}) = \operatorname{rk}(\mathbf{K}/F_k)c_1(\mathbf{K}'/F_k) - \operatorname{rk}(\mathbf{K}'/F_k)c_1(\mathbf{K}/F_k).$$

By Remark 6.12, this is strictly positive if $\mathbf{K}'/F_k$ is of type (1), (2) or (7) contradicting the semistability of $\mathbf{K}$. Thus $\mathbf{K}/F_k$ must contain a complex F_{k+1}/F_k of type (3). Now apply induction. $\square$

There is one possibility that does not occur in [DL] and is more difficult to treat. This is when $\mathbf{K}$ is a complex of the form (6.4) for which $\operatorname{coker}(B)$ has a composition series of fat points. Here, a *fat point in* qgr-S is a simple object $\mathcal{M}$ whose module of global sections $\Gamma^*(\mathcal{M})$ has Hilbert series $n(1-t)^{-1}$ for some $n > 1$. Fat points do not belong to qgr-S/gS. The next few lemmas are concerned with this case.

Lemma 6.15. *Let $S \in \underline{\mathrm{AS}}_3'$ and assume that we have an exact sequence*

$$\mathcal{O}_S^a \xrightarrow{\alpha} \mathcal{O}_S^b(1) \xrightarrow{\beta} \mathcal{K} \longrightarrow 0. \tag{6.10}$$

for which $\mathcal{K} \neq 0$ but $\beta|_E = 0$.

(1) *If $K = \Gamma^*(\mathcal{K})$ then GKdim$(K) = 1$ and $K_n g = K_{n+3}$, for all $n \gg 0$.*
(2) *Let $\beta(t) : \mathcal{O}^b(t+1) \to \mathcal{K}(t)$ be the natural map induced by β for $t \in \mathbb{Z}$. Then $\dim_k \operatorname{Im} \mathrm{H}^0(\beta(t)) \geq \dim_k \operatorname{Im} \mathrm{H}^0(\beta(-1))$ for all $t \geq 0$.*

Proof. (1) By right exactness, $\mathcal{K}|_E = 0$. Thus if $K = \Gamma^*(\mathcal{K})$, then K/Kg is finite dimensional and so $K_n = K_{n-3}g$ for all $n \gg 0$.

(2) Break (6.10) into two exact sequences:

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{O}_S^a \xrightarrow{\alpha} \mathcal{B} \longrightarrow 0. \tag{6.11}$$

and

$$0 \longrightarrow \mathcal{B} \xrightarrow{\iota} \mathcal{O}_S^b(1) \xrightarrow{\beta} \mathcal{K} \longrightarrow 0. \tag{6.12}$$

Taking cohomology of (6.12) gives the exact sequence

$$\mathrm{H}^0(\mathcal{O}^b(1+u)) \xrightarrow{\mathrm{H}^0(\beta(u))} \mathrm{H}^0(\mathcal{K}(u)) \to \mathrm{H}^1(\mathcal{B}(u)) \to 0 \qquad \text{for} \quad u \in \mathbb{Z}.$$

By [ATV2, Proposition 6.6(iv)] and Lemma 2.3(3), $h^0(\mathcal{K}(u)) = \dim \mathrm{H}^0(\mathcal{K}(u))$ is constant for $u \in \mathbb{Z}$. Thus, by part (1), in order to prove the lemma it suffices to show that $h^1(\mathcal{B}(-1)) \geq h^1(\mathcal{B}(t))$, for $t \geq 0$.

By Lemma 2.3(5), $\mathrm{H}^j(\mathcal{O}^a(u)) = 0$ for $j = 1, 2$ and $u \geq -1$. Thus, taking cohomology of shifts of (6.11) shows that $\mathrm{H}^1(\mathcal{B}(u)) \cong \mathrm{H}^2(\mathcal{A}(u))$ for $u \geq -1$. So it suffices to prove that $h^2(\mathcal{A}(-1)) \geq h^2(\mathcal{A}(t))$ for $t \geq 0$. Finally, Serre duality (Proposition 2.4) gives

$$h^2(\mathcal{A}(t)) = \dim_k \mathrm{Hom}(\mathcal{A}, \mathcal{O}(-3-t)) \quad \leq \quad \dim_k \mathrm{Hom}(\mathcal{A}, \mathcal{O}(-2)) = h^2(\mathcal{A}(-1)),$$

for $t \geq 0$. □

An S-module M is called 1-*homogeneous* if $\mathrm{GKdim}\, M = 1$ and M has no finite-dimensional submodules.

Lemma 6.16. *Let Q be a graded 1-homogeneous right S-module that is generated by a t-dimensional subspace of Q_1. Then the minimal projective resolution of Q has the form*

$$0 \longrightarrow P_2 \xrightarrow{\theta} P_1 \xrightarrow{\phi} S(1)^t \longrightarrow Q \longrightarrow 0, \tag{6.13}$$

such that the following holds:

(1) *$P_2 \cong S^s(-1) \oplus P_2'$ where $P_2' \cong \bigoplus_m S(j_m)$ and each $j_m < -1$;*
(2) *$P_1 \cong S^r \oplus P_1'$ where $P_1' \cong \bigoplus_m S(k_m)$ and each $k_m < 0$;*
(3) *$r \leq 2t$ and $\theta(S^s(-1)) \subseteq S^r$.*

Proof. [ATV2, Proposition 2.46] implies that $\mathrm{hd}(Q) \leq 2$ and so Q has a minimal resolution of the form (6.13) satisfying (1) and (2). By Lemma 6.15, $\dim Q_2 \geq t$ and so computing the dimension of (6.13) in degree 2 shows that $r \leq 3t - \dim Q_2 \leq 2t$. It follows easily from the minimality of (6.13) that $\theta(S^s(-1)) \subseteq S^r$. □

Proposition 6.17. *Suppose that*

$$\mathbf{K}: \quad \mathcal{O}(-1)^a \xrightarrow{\theta} \mathcal{O}^b \xrightarrow{\phi} \mathcal{O}(1)^c \tag{6.14}$$

is a Kronecker complex with θ injective and $\mathrm{GKdim}(\Gamma^*\mathrm{coker}(\phi)) \leq 1$. *Then either* $\mathrm{coker}(\phi) = 0$ *or there exists a nonzero quotient complex of* $\mathbf{K}$ *of the form*

$$\mathbf{K}': \quad \mathcal{O}(-1)^s \xrightarrow{\theta'} \mathcal{O}^r \xrightarrow{\phi'} \mathcal{O}(1)^t \tag{6.15}$$

with $s + t \leq r \leq 2t$.

Proof. Assume that $\operatorname{coker}(\phi) \neq 0$ and pick a nonzero, 1-homogeneous factor Q of $\Gamma^*(\operatorname{coker}(\phi))$. Then Q is generated by a subspace of Q_1 of dimension $t \leq c$. By Lemma 6.16 there is a resolution of the form (6.13), the image of which in qgr-S has the form

$$\mathbf{R}:\ 0 \to \mathcal{O}(-1)^{s'} \oplus \pi P_2' \xrightarrow{\theta'} \mathcal{O}^{r'} \oplus \pi P_1' \xrightarrow{\phi'} \mathcal{O}(1)^t \xrightarrow{\psi'} \pi Q \to 0$$

where the P_i' are generated in degrees $\geq i$. By construction, the map $S(1)^c \twoheadrightarrow Q$ factors through $S(1)^t$. We next show that this induces a map $\mathbf{K} \to \mathbf{R}$; more precisely, we construct a commutative diagram of the form

$$\begin{tikzcd}
0 \arrow[r] & S(-1)^a \arrow[r, "\theta"] \arrow[d, "\alpha_2"'] & S^b \arrow[r, "\phi"] \arrow[d, "\alpha_1"'] & S(1)^c \arrow[r, "\psi"] \arrow[d, "\alpha_0"'] & Q \arrow[r] \arrow[d, "\alpha_{-1}"'] & 0 \\
0 \arrow[r] & S(-1)^{s'} \oplus P_2' \arrow[r, "\theta'"] & S^{r'} \oplus P_1' \arrow[r, "\phi'"] & S(1)^t \arrow[r, "\psi'"] & Q \arrow[r] & 0
\end{tikzcd}$$

whose image in qgr-S is the desired map $\mathbf{K} \to \mathbf{R}$. The maps α_0 and α_{-1} are the surjections already defined and so the final square is commutative. Since the second row of this diagram is exact, the usual diagram chase constructs α_1 and α_2.

By the definition of the P_i', one has $\alpha_1(S^b) \subseteq S^{r'}$ and $\alpha_2(S(-1)^a) \subseteq S(-1)^{s'}$. Thus, if $\mathbf{K}'$ is the image of $\mathbf{K}$ in $\mathbf{R}$, then $\mathbf{K}'$ is a Kronecker complex of the the form

$$\mathcal{O}(-1)^s \xrightarrow{\theta'} \mathcal{O}^r \xrightarrow{\phi'} \mathcal{O}(1)^t,$$

where $r = \operatorname{rk}(\alpha_1(\mathcal{O}^b))$ and $s = \operatorname{rk}(\alpha_1(\mathcal{O}(-1)^a))$. The map θ' is injective since it is induced from the injection θ. Moreover, $r \leq r' \leq 2t$ by Lemma 6.16(3).

It remains to prove that $s + t \leq r$. Since this is just a question of ranks, we may pass to the (graded or ungraded) division ring of fractions D of S. Then $\mathbf{K}' \otimes_S D$ gives a complex $D^s \hookrightarrow D^r \xrightarrow{\phi' \otimes D} D^t$. By hypothesis, $\phi \otimes D$ is surjective and hence so is $\phi' \otimes D$. This gives the desired inequality. □

We are now ready to prove Proposition 6.9.

Proof of Proposition 6.9. Recall that we are given a semistable, normalized Kronecker complex of the form

$$\mathbf{K}: \qquad \mathcal{O}(-1) \otimes V_{-1} \xrightarrow{A} \mathcal{O} \otimes V_0 \xrightarrow{B} \mathcal{O}(1) \otimes V_1,$$

over an algebra $S \in \underline{\mathrm{AS}}_3$ and we wish to prove that $\mathbf{K}$ is a torsion-free monad. As was remarked earlier, since the proposition is a result about qgr-S, we may assume that $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}_3'$.

Choose a filtration $F_0 \subset \cdots \subset F_\ell \subset \mathbf{K}$ by Lemma 6.14. By Remark 6.11, $\mathrm{H}^{-1}(F_i/F_{i-1}) = 0$ for each i and so $\mathrm{H}^{-1}(F_\ell) = 0$. Next, consider $\ker(A_{\mathbf{K}/F_\ell})$. We have an exact sequence

$$0 \to \ker(A_{\mathbf{K}/F_\ell}) \to \mathcal{O}(-1) \otimes V_{-1}' \xrightarrow{A_{\mathbf{K}/F_\ell}} \operatorname{Im}(A_{\mathbf{K}/F_\ell}) \to 0,$$

and $\operatorname{Im}(A_{\mathbf{K}/F_\ell})$ is torsion-free since it is a submodule of a vector bundle. By Lemmas 2.6 and 6.14(iii), this implies that $\ker(A_{\mathbf{K}/F_\ell})|_E = \ker(A_{\mathbf{K}/F_\ell}|_E) = 0$. Thus $\ker(A_{\mathbf{K}/F_\ell})$ is a torsion submodule of $\mathcal{O}(-1) \otimes V_{-1}'$ and so it is zero. Therefore, $A_{\mathbf{K}/F_\ell}$ and A_{F_ℓ} are both injective, and so $A_{\mathbf{K}}$ is also injective.

We next show that B is surjective. By Corollary 6.13 and Lemma 6.10(a), $B|_E$ is surjective. Suppose that $B_{\mathbf{K}}$ is not surjective. Then Lemma 6.15(1) implies that $\operatorname{GKdim} \Gamma^*(\operatorname{Coker}(B)) = 1$ and so, by Proposition 6.17, there is a quotient complex

$\mathbf{K}'$ of $\mathbf{K}$ of the form $\mathcal{O}(-1)^{t-a-b} \to \mathcal{O}^{2t-a} \to \mathcal{O}(1)^t$ with $a, b \geq 0$ and $t > 0$. By definition, $c_1(\mathbf{K}') = t - a - b - t = -a - b$ and $\mathrm{rk}(\mathbf{K}') = b$. This gives

$$\mathrm{rk}(\mathbf{K})c_1(\mathbf{K}') - \mathrm{rk}(\mathbf{K}')c_1(\mathbf{K}) = -a\,\mathrm{rk}(\mathbf{K}) - b\,\{\mathrm{rk}(\mathbf{K}) + c_1(\mathbf{K})\} = x,$$

say. Since $\mathbf{K}$ is normalized, $x \leq 0$ with equality if and only if $a = b = 0$. So $\mathbf{K}'$ de-semistabilizes $\mathbf{K}$ unless $a = b = 0$, in which case we find that

$$\mathrm{rk}(\mathbf{K})\chi(\mathbf{K}') - \mathrm{rk}(\mathbf{K}')\chi(\mathbf{K}) = \mathrm{rk}(\mathbf{K}')(t - a - 3t) < 0$$

and so $\mathbf{K}'$ still de-semistabilizes $\mathbf{K}$. This contradiction shows that B is surjective.

Finally, we must show that $\mathrm{H}^0(\mathbf{K})$ is torsion-free. Write $\mathbf{K}/F_\ell$ as the complex

$$0 \to \mathcal{O}(-1) \otimes V'_{-1} \xrightarrow{A'} \mathcal{O} \otimes V'_0 \xrightarrow{B'} \mathcal{O}(1) \otimes V'_1 \to 0. \tag{6.16}$$

We first want to show that $\mathrm{coker}(A')$ is torsion-free, so assume that its torsion submodule $\mathcal{T}$ is nonzero. By the earlier part of the proof, A' is injective. Thus, by Lemma 2.3(5), applying $\Gamma(-)$ to (6.16) gives the exact sequence

$$0 \to S(-1)^a \xrightarrow{A''} S^b \xrightarrow{B''} C \to 0, \tag{6.17}$$

where $A'' = \Gamma^*(A')$ and $C = \mathrm{coker}(A'')$. Let T denote the torsion submodule of C. By Lemma 2.3(2), $\pi(C) = \mathrm{coker}(A')$ and $\pi(T) = \mathcal{T}$, so $T \neq 0$. The sequence (6.17) shows that C has projective dimension $\mathrm{hd}\, C \leq 1$. Since C/T has no finite dimensional submodules, [ATV2, Proposition 2.46(i)] implies that $\mathrm{hd}(C/T) \leq 2$ and hence that $\mathrm{hd}(T) \leq 1$. By [ATV2, Theorem 4.1(iii)], $\mathrm{GKdim}\, T = 2$.

By Lemma 6.14(iii), $A'|_E$ is injective and hence so is $A'' \otimes_S (S/gS)$. Thus $\underline{\mathrm{Tor}}_1^S(C, S/gS) = 0$. But this Tor group is also $\mathrm{Ker}\,[C \otimes Sg \to C]$. Therefore multiplication by g is injective on C and its submodule T. By [Zh, Lemma 2.2(3)], this implies that $\mathrm{GKdim}\, T/Tg = \mathrm{GKdim}\, T - 1 = 1$. On the other hand, as C/T is torsion-free, the argument in Lemma 2.6 shows that $\underline{\mathrm{Tor}}_1^S(C/T, S/gS) = 0$. Therefore, $T/Tg \hookrightarrow C/Cg$ and so $\mathcal{T}/\mathcal{T}g = \pi(T/Tg)$ is submodule of $\pi(C/Cg) = \mathrm{coker}(A')|_E$ of finite length. This contradicts Lemma 6.14(iii).

Thus $\mathrm{coker}(A')$ is torsion-free. By the definition of a type (3) complex in Lemma 6.10(a), $\mathrm{coker}(A_{F_i/F_{i-1}})$ is torsion-free for each i. Thus $\mathrm{coker}(A_{\mathbf{K}})$ is an iterated extension of torsion-free modules and hence is torsion-free. Therefore its submodule $\mathrm{H}^0(\mathbf{K})$ is torsion-free. This completes the proof of Proposition 6.9. □

It is now easy to generalize Proposition 6.9 to families of Kronecker complexes, defined as follows.

Definition 6.18. Let R be a commutative noetherian k-algebra and $S \in \underline{\mathrm{AS}}_3$. Recall that $S_R = S \otimes_k R$, regarded as a graded R-algebra. A *family of Kronecker complexes* parametrized by a scheme $\mathrm{Spec}\, R$, also called a Kronecker complex in qgr-S_R, is a complex of the form

$$\mathbf{K}: \quad \mathcal{O}(-1) \otimes_R V_{-1} \xrightarrow{A} \mathcal{O} \otimes_R V_0 \xrightarrow{B} \mathcal{O}(1) \otimes_R V_1 \tag{6.18}$$

in qgr-S_R where V_{-1}, V_0 and V_1 are finite-rank vector bundles on $\mathrm{Spec}\, R$. The definition of families of (geometrically, semi)stable Kronecker complexes follows from Definition 5.3. As before, a family of geometrically (semi)stable Kronecker complexes is automatically (semi)stable.

As usual, morphisms of Kronecker complexes in qgr-S_R are morphisms of complexes. Two such complexes (or modules) $\mathbf{K}$ and $\mathbf{K}'$ are *equivalent* if there is a line bundle M on $\mathrm{Spec}\, R$ such that $\mathbf{K} \otimes_R M$ and $\mathbf{K}'$ are isomorphic in qgr-S_R.

Corollary 6.19. *Suppose that* $\mathbf{K}$ *is a semistable Kronecker complex in* qgr-S_R. *Then* $\mathbf{K}$ *is a torsion-free monad in* qgr-S_R *in the sense of Definition 5.2.*

Proof. By definition, if $p \in \operatorname{Spec} R$, then $\mathbf{K} \otimes k(p)$ is semistable and so Proposition 6.9 implies that $\mathbf{K} \otimes k(p)$ is a torsion-free monad. □

We end the section by checking that the notions of semistability for monads and for their cohomology do correspond.

Proposition 6.20. *Let* $S \in \underline{\mathrm{AS}}_3$ *and let* R *be a commutative noetherian* k*-algebra. Then the map* $\mathbf{K} \mapsto \mathrm{H}^0(\mathbf{K})$ *induces an equivalence of categories between:*

(i) *the category of geometrically (semi)stable normalized Kronecker complexes* $\mathbf{K}$ *in* qgr-S_R *and*
(ii) *the category of* R*-flat families* $\mathcal{M}$ *of geometrically (semi)stable, normalized torsion-free modules in* qgr-S_R.

Proof. By Corollary 6.19, the Kronecker complex $\mathbf{K}$ in (i) is a torsion-free monad in $\mathrm{Monad}(S_R)$, while Lemma 6.4 implies that $\mathcal{M} \in (\text{qgr-}S_R)_{\mathrm{VC}}$ in part (ii). Now consider the equivalence of categories $\mathrm{Monad}(S_R) \simeq (\text{qgr-}S_R)_{\mathrm{VC}}$ from Theorem 5.8(1). By definition the torsion-free objects correspond under this equivalence and it is elementary that the normalized objects also correspond. Thus in order to prove the proposition, it remains to check that the geometrically (semi)stable objects correspond. If $\mathcal{M} \in (\text{qgr-}S_R)_{\mathrm{VC}}$, then Theorem 5.8(2) implies that $\mathbf{K}(\mathcal{M}) \otimes F = \mathbf{K}(\mathcal{M} \otimes F)$ for every geometric point $\operatorname{Spec} F \to \operatorname{Spec} R$ and so we need only prove that the (semi)stability of $\mathcal{M} \otimes F$ is equivalent to that of $\mathbf{K}(\mathcal{M} \otimes F)$. In other words, after replacing k by F, it remains to prove that $\mathcal{M} \in \text{qgr-}S_k$ is (semi)stable if and only if $\mathbf{K} = \mathbf{K}(\mathcal{M})$ is (semi)stable.

If $\mathbf{K}$ is (semi)stable then the proof of [DL, Proposition 2.3(3)] can be used, essentially without change, to prove that $\mathcal{M} = \mathrm{H}^0(\mathbf{K})$ is (semi)stable.

The other direction does not quite follow from the corresponding result in [DL] and before proving it we need a definition and a lemma. Let $\mathbf{K}$ denote a Kronecker complex over $S \in \underline{\mathrm{AS}}_3$. A *maximal subcomplex* of $\mathbf{K}$ is a subcomplex $\mathbf{K}'$ that realizes the maximum of $\mathrm{rk}(\mathbf{K})p_{\mathbf{K}'} - \mathrm{rk}(\mathbf{K}')p_{\mathbf{K}}$ among all subcomplexes of $\mathbf{K}$.

Sublemma 6.21. ([DL, Lemme 2.4]) *Suppose that* $\mathbf{K}$ *is a normalized monad over* $S \in \underline{\mathrm{AS}}'_3$. *If* $\mathbf{K}'$ *denotes a maximal subcomplex of* $\mathbf{K}$, *then* $\mathrm{H}^1(\mathbf{K}'|_E) = 0$.

Proof. By Lemma 6.10, it suffices to check that $\mathbf{K}'$ has no quotient complex $\mathbf{L}$ of types (3–6). Suppose that such an $\mathbf{L}$ exists and set $\mathbf{L}' = \mathrm{Ker}(\mathbf{K}' \to \mathbf{L})$. Then

$$\mathrm{rk}(\mathbf{K})p_{\mathbf{L}'} - \mathrm{rk}(\mathbf{L}')p_{\mathbf{K}} = \{\mathrm{rk}(\mathbf{K})p_{\mathbf{K}'} - \mathrm{rk}(\mathbf{K}')p_{\mathbf{K}}\} - \{\mathrm{rk}(\mathbf{K})p_{\mathbf{L}} - \mathrm{rk}(\mathbf{L})p_{\mathbf{K}}\}$$

by the additivity of ranks and Hilbert polynomials. Remark 6.12 shows that $\mathrm{rk}(\mathbf{K})p_{\mathbf{L}} - \mathrm{rk}(\mathbf{L})p_{\mathbf{K}} < 0$. This contradicts the maximality of $\mathbf{K}'$. □

We now return to the proof of Proposition 6.20. By the first paragraph of the proof it remains to show that, if $\mathcal{M} \in \text{qgr-}S$ is (semi)stable, then so is $\mathbf{K} = \mathbf{K}(\mathcal{M})$.

As usual, we may assume that $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$ and we write $\mathbf{K}$ as

$$\mathcal{O}(-1) \otimes V_{-1} \xrightarrow{A} \mathcal{O} \otimes V_0 \xrightarrow{B} \mathcal{O}(1) \otimes V_1.$$

If $\mathbf{K}'$ is a maximal subcomplex of $\mathbf{K}$ then Sublemma 6.21 implies that $\mathrm{H}^1(\mathbf{K}'|_E) = 0$. Let $\mathbf{K}'' = \mathbf{K}/\mathbf{K}'$. We first prove that $\mathrm{H}^{-1}(\mathbf{K}'') = 0$. There is a filtration

$$\mathbf{K}' = F_0 \subset F_1 \subset \cdots \subset F_l \subseteq \mathbf{K}$$

of $\mathbf{K}$ with each F_i/F_{i-1} of type (3), such that $\mathbf{K}/F_l$ contains no subcomplex of type (3). In particular, $c_1(F_l) = c_1(\mathbf{K}')$, by Remark 6.12.

We claim that $\mathbf{K}/F_l$ contains no subcomplex of type (1), (2), or (7). Suppose there were such a subcomplex, say $\mathbf{L}/F_l$. Then Remark 6.12 implies that $c_1(\mathbf{L}) = c_1(F_l) + 1 = c_1(\mathbf{K}') + 1$. Thus

$$\mathrm{rk}(\mathbf{K})c_1(\mathbf{L}) - \mathrm{rk}(\mathbf{L})c_1(\mathbf{K}) = \mathrm{rk}(\mathbf{K})c_1(\mathbf{K}') - \mathrm{rk}(\mathbf{K}')c_1(\mathbf{K}) + [\mathrm{rk}(\mathbf{K}) - c_1(\mathbf{K})\,\mathrm{rk}(\mathbf{L}/F_l)]\,.$$

In the three cases, Remark 6.12 implies that $\mathrm{rk}(\mathbf{L}/F_l) = -1, 0, 1$, respectively. This in turn implies that $[\mathrm{rk}(\mathbf{K}) - c_1(\mathbf{K})\,\mathrm{rk}(\mathbf{L}/F_l)] > 0$ (when $\mathrm{rk}(\mathbf{L}/F_l) = -1$ this needs the fact that $\mathbf{K}$ is normalized). By the displayed equation, this contradicts the maximality of $\mathbf{K}'$ and proves the claim.

By Lemma 6.10, the last paragraph implies that $\mathrm{H}^{-1}((\mathbf{K}/F_l)|_E) = 0$. Applying Lemma 2.6 to the short exact sequence

$$0 \to \mathrm{H}^{-1}(\mathbf{K}/F_l) \to (\mathbf{K}/F_l)_{-1} \xrightarrow{A_{\mathbf{K}/F_l}} \mathrm{Im}(A_{\mathbf{K}/F_l}) \to 0$$

shows that $\mathrm{H}^{-1}(\mathbf{K}/F_l) = 0$. But $\mathrm{H}^{-1}(F_l/\mathbf{K}') = 0$ by construction, so $\mathrm{H}^{-1}(\mathbf{K}'') = \mathrm{H}^{-1}(\mathbf{K}/\mathbf{K}') = 0$, as desired.

It follows that the map $\mathrm{H}^0(\mathbf{K}') \to \mathrm{H}^0(\mathbf{K}) = M$ is injective. Let $C = \mathrm{coker}(B_{\mathbf{K}'})$. As $A_{\mathbf{K}'}$ is injective, $p_{\mathrm{H}^0(\mathbf{K}')} = p_{\mathbf{K}'} + p_C \geq p_{\mathbf{K}'}$. By Sublemma 6.21, C must be torsion and hence have rank zero. In particular, $\mathrm{rk}(\mathrm{H}^0(\mathbf{K}')) = \mathrm{rk}(\mathbf{K}')$. Since the rank of a monad equals the rank of its cohomology, these observations imply that

$$\mathrm{rk}(\mathbf{K})p_{\mathbf{K}'} - \mathrm{rk}(\mathbf{K}')p_{\mathbf{K}} \;\leq\; \mathrm{rk}(M)p_{\mathrm{H}^0(\mathbf{K}')} - \mathrm{rk}(\mathrm{H}^0(\mathbf{K}'))p_M.$$

Therefore, the (semi)stability of M implies (semi)stability of $\mathbf{K}$, completing the proof of Proposition 6.20. □

7. Moduli Spaces: Construction

In this section we use a Grassmannian embedding to construct a projective moduli space of semistable modules in qgr-S for $S \in \underline{\mathrm{AS}}_3$ (see Theorem 7.10) and determine cases when the moduli space is fine (see Proposition 7.15 and its corollaries). This will complete the proof of Theorems 1.6 and 1.5 from the introduction. The more subtle properties of these moduli spaces (notably, determining cases where they are smooth or connected) will be examined in Section 8.

The idea behind the proofs of this section is similar to the classical case: in Subsection 7.1 we prove that the moduli functor of framed semistable Kronecker complexes (semistable Kronecker complexes that have been appropriately rigidified) can be embedded in a product of Grassmannians GR, and that the image is the semistable locus for the natural group action on a closed subscheme of GR. This allows us to prove in Subsection 7.2 that the GIT quotient of this subscheme is exactly the moduli space we wish to construct. The description in terms of Kronecker complexes then gives a convenient tool for proving smoothness and existence of universal modules.

We remind the reader of the relevant definitions. Let F be a contravariant functor from the category of (typically, noetherian affine) k-schemes to the category of sets. Then a scheme Y together with a morphism of functors $F \to h_Y = \mathrm{Hom}_{k\text{-Sch}}(-, Y)$ *corepresents* F if this morphism is universal for morphisms $F \to h_X$ from F to schemes X. In this case Y is also called a *coarse moduli space for* F. The scheme Y

represents F, equivalently is a *fine moduli space for* F, if $F = h_Y$ under the usual embedding of the category of schemes into the category of contravariant functors.

We are interested in classifying torsion-free modules over $S \in \underline{\mathrm{AS}}_3$ and by shifting there is no harm in assuming that they are normalized (see Remark 7.9 for the formal statement). By Proposition 6.20 this means that we can work with normalized Kronecker complexes over $S \in \underline{\mathrm{AS}}_3$. Thus, through the end of Subsection 7.1 we can and will fix the following data.

Notation 7.1. *Fix integers $r \geq 1$, c_1 and χ with $-r < c_1 \leq 0$. A normalized Kronecker complex with rank r, first Chern class c_1 and Euler characteristic χ will be called* a Kronecker complex with invariants $\{r, c_1, \chi\}$. *In order to avoid excessive subscripts it will typically be written as*

$$K \otimes \mathcal{O}(-1) \xrightarrow{A} H \otimes \mathcal{O} \xrightarrow{B} L \otimes \mathcal{O}(1), \tag{7.1}$$

where $\dim K = d_{-1}$, $\dim H = n$ *and* $\dim L = d_1$. *These numbers are determined by the other invariants and so they too will be fixed throughout the section. Specifically:*

$$d_{-1} = 2c_1 + r - \chi, \qquad n = 3r + 3c_1 - 2\chi \quad \textit{and} \quad d_1 = c_1 + r - \chi. \tag{7.2}$$

Finally, the vector space H of dimension n will also be fixed throughout.

Let R be a commutative k-algebra. A complex in qgr-S_R is a *framed Kronecker complex* $(\mathbf{K}, \phi)$ if it is a Kronecker complex of the form (6.18) with a specific choice of isomorphism $\phi : V_0 \cong H_R = R \otimes H$. Equivalently, it has the form

$$K \otimes_R \mathcal{O}_{S_R}(-1) \xrightarrow{A} H_R \otimes_R \mathcal{O}_{S_R} \xrightarrow{B} L \otimes_R \mathcal{O}_{S_R}(1), \tag{7.3}$$

for some bundles K and L. It is implicit in this description that isomorphisms of framed Kronecker complexes restrict to the identity on $H_R \otimes \mathcal{O}_{S_R}$.

In the notation of Definitions 6.18 and 5.3, let $\mathcal{K}^{ss}$ denote the moduli functor for equivalence classes of families of geometrically semistable Kronecker complexes in qgr-S with invariants $\{r, c_1, \chi\}$. Write $\mathcal{K}^s$ for its subfunctor of geometrically stable Kronecker complexes. Let $\widehat{\mathcal{K}}^{ss}$ denote the moduli functor of families of framed geometrically semistable Kronecker complexes with invariants $\{r, c_1, \chi\}$ with subfunctor $\widehat{\mathcal{K}}^s$ of geometrically stable framed Kronecker complexes. The group $\mathrm{GL}(H)$ acts on $\widehat{\mathcal{K}}^{ss}$: an element $g \in \mathrm{GL}(H)$ takes a complex $\mathbf{K}$ as in (7.3) to a complex $g \cdot \mathbf{K}$ with the same objects but with the maps A and B replaced by $(g \otimes 1) \circ A$ and $B \circ (g^{-1} \otimes 1)$.

7.1. **Semistability via Grassmannians.** In this subsection, we convert the problem of classifying Kronecker complexes into a problem about Grassmannians. We begin with a general framework, since this will also enable us to describe our moduli spaces for families of AS regular algebras as well as for individual algebras.

Let $\mathcal{B}$ be a k-scheme. We define $S = S_{\mathcal{B}}$ to be *a $\mathcal{B}$-flat family of algebras in* $\underline{\mathrm{AS}}_3$ if S is a flat, connected graded $\mathcal{B}$-algebra such that $S_{\mathcal{B}} \otimes k(p) \in \underline{\mathrm{AS}}_3(k(p))$ for all points $p \in \mathcal{B}$. If $U = \operatorname{Spec} R \xrightarrow{f} \mathcal{B}$ is an affine $\mathcal{B}$-scheme, then we write $S_U = S_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}_U$ for the corresponding family of $\mathcal{O}_U$-algebras.

We will be interested in subschemes of the following product of Grassmannians:

$$\mathrm{GR}_{\mathcal{B}} = \mathrm{Gr}_{d_{-1}}(H_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}}} S_1) \times \mathrm{Gr}^{d_1}(H_{\mathcal{B}} \otimes_{\mathcal{O}_{\mathcal{B}}} S_1^*), \tag{7.4}$$

where $\mathrm{Gr}_{d_{-1}}$ denotes the relative Grassmannian of rank d_{-1} $\mathcal{O}_{\mathcal{B}}$-subbundles of $H_{\mathcal{B}} \otimes S_1$ and Gr^{d_1} denotes the relative Grassmannian of rank d_1 quotient bundles of $H_{\mathcal{B}} \otimes S_1^*$. As in [HL, Example 2.2.3], $\mathrm{GR}_{\mathcal{B}}$ represents the functor that, to a $\mathcal{B}$-scheme $U \xrightarrow{f} \mathcal{B}$, associates the set of pairs (called *KL-pairs*)

$$\big(K \overset{i}{\hookrightarrow} H_U \otimes_{\mathcal{O}_U} (f^*S_1),\ H_U \otimes_{\mathcal{O}_U} (f^*S_1)^* \overset{\pi}{\twoheadrightarrow} L\big) \tag{7.5}$$

of a subbundle K of $H_U \otimes S_1$ of rank d_{-1} and a quotient bundle L of $H_U \otimes S_1^*$ of rank d_1 on U. In particular, by the Yoneda Lemma, the identity map $\mathrm{GR}_{\mathcal{B}} \to \mathrm{GR}_{\mathcal{B}}$ determines a KL-pair

$$\mathcal{P} = (\overline{K} \xrightarrow{i_{\mathcal{P}}} S_1 \otimes H_{\mathrm{GR}},\ S_1^* \otimes H_{\mathrm{GR}} \xrightarrow{\pi_{\mathcal{P}}} \overline{L})$$

on $\mathrm{GR}_{\mathcal{B}}$ that is *universal* in the sense that a KL-pair on U is the pullback $\tilde{f}^*\mathcal{P}$ along a unique map $U \xrightarrow{\tilde{f}} \mathrm{GR}_{\mathcal{B}}$. We will typically suppress the pull-back maps f^* or $\tilde{f}^*$ when it causes no confusion.

Remark 7.2. Let $\mathcal{H}$ be a vector bundle of rank n on a k-scheme $\mathcal{B}$, respectively $\mathcal{H} = H_{\mathcal{B}}$ and let

$$P = (K \subset \mathcal{H}_U \otimes S_1, \mathcal{H}_U \otimes S_1^* \to L)$$

be a vector subbundle and quotient bundle over $U = \operatorname{Spec} R$. Then one obtains a diagram of the form (6.18), respectively (7.3), by taking for A the composite map

$$A:\ K \otimes \mathcal{O}_{S_U}(-1) \to \mathcal{H}_U \otimes S_1 \otimes \mathcal{O}_{S_U}(-1) \to \mathcal{H}_U \otimes \mathcal{O}_{S_U},$$

(where the final map comes from multiplication in S_U and all tensor products are over R) and for B the composite map

$$B:\ \mathcal{H}_U \otimes \mathcal{O}_{S_U} \to \mathcal{H}_U \otimes S_1^* \otimes S_1 \otimes \mathcal{O}_{S_U} \to L \otimes S_1 \otimes \mathcal{O}_{S_U} \to L \otimes \mathcal{O}_{S_U}(1).$$

Let $\mathbf{K}(P)$ denote the diagram associated to P. Since $\mathbf{K}(P)$ need not be a complex, we define $N_{\mathcal{B},\mathcal{H}}$ to be the subfunctor of the relative Grassmannian functor $\mathrm{GR}_{\mathcal{B},\mathcal{H}}$ consisting of those P for which the associated diagram $\mathbf{K}(P)$ is a complex. When $\mathcal{H} = H_{\mathcal{B}}$, write $N_{\mathcal{B},H_{\mathcal{B}}} = N_{\mathcal{B}}$; then the above construction defines a map from $N_{\mathcal{B}}$ to the functor of isomorphism classes of framed Kronecker complexes.

Remark 7.3. Given a vector space W, $\mathrm{GL}(H)$ acts on the Grassmannian $\mathrm{Gr} = \mathrm{Gr}_d(W \otimes H)$ as follows: $g \in \mathrm{GL}(H)$ takes a subspace $V \subset W \otimes H$ to $gV = (1 \otimes g) \cdot V$ and the quotient H/V to $H/(1 \otimes g)V$. Since this action is defined via the subspaces themselves it gives a $\mathrm{GL}(H)$-equivariant structure on the universal subbundle and universal quotient. Note that if g is in the centre of $\mathrm{GL}(H)$ then $gV = V$ but g acts by scalar multiplication m_g on V and H/V.

More globally, let $\mathcal{V} \xrightarrow{\iota} \mathcal{O}_{\mathrm{Gr}} \otimes W \otimes H$ denote the universal subbundle on the Grassmannian. The composition

$$\widetilde{\iota} : \mathcal{V} \xrightarrow{\iota} \mathcal{O}_{\mathrm{Gr}} \otimes W \otimes H \xrightarrow{1 \otimes g} \mathcal{O}_{\mathrm{GL}(H) \times \mathrm{Gr}} \otimes W \otimes H$$

has image $g\mathcal{V}_x$ in the fibre over $x \in \mathrm{Gr}$ and so the map $m_g : \mathrm{Gr} \to \mathrm{Gr}$ is the one determined by the subbundle $\mathrm{Im}(\widetilde{\iota})$. Conversely, if $\mathcal{O}_{\mathrm{Gr}} \otimes W \otimes H \xrightarrow{\pi} \mathcal{Q}$ is the universal quotient bundle (that is, the cokernel of ι), then the composite map

$$\mathcal{O}_{\mathrm{Gr}} \otimes W \otimes H \xrightarrow{1 \otimes g^{-1}} \mathcal{O}_{\mathrm{Gr}} \otimes W \otimes H \xrightarrow{\pi} \mathcal{Q}$$

has the same kernel as $\widetilde{\iota}$, so it also corresponds to $m_g : \mathrm{Gr} \to \mathrm{Gr}$.

Now let $\mathrm{GL}(H)$ act diagonally on $\mathrm{GR}_{\mathcal{B}}$. Then the last paragraph implies that $m_g^*\mathcal{P}$ is isomorphic to the pair given by the maps $\big((1\otimes m_g)\circ i_{\mathcal{P}}, \pi_{\mathcal{P}}\circ(1\otimes m_{g^{-1}})\big)$. Thus $\mathbf{K}(m_g^*\mathcal{P}) \cong g\cdot \mathbf{K}(\mathcal{P})$. In particular, the subfunctor $N_{\mathcal{P}}$ is $\mathrm{GL}(H)$-invariant and the map from $N_{\mathcal{B}}$ to the functor of isomorphism classes of framed Kronecker complexes is $\mathrm{GL}(H)$-equivariant.

Lemma 7.4. *Let $S = S_{\mathcal{B}}$ be a flat family of algebras in $\underline{\mathrm{AS}}_3$ parametrized by a k-scheme $\mathcal{B}$. Then:*

(1) *There is a closed $\mathrm{GL}(H)$-invariant subscheme $\mathcal{N}_{\mathcal{B}} \subset \mathrm{GR}_{\mathcal{B}}$ that represents the subfunctor $N_{\mathcal{B}}$.*
(2) *If $\mathcal{B}' \to \mathcal{B}$ is a morphism of k-schemes, then $\mathcal{N}_{\mathcal{B}'} = \mathcal{N}_{\mathcal{B}} \times_{\mathcal{B}} \mathcal{B}'$.*

Proof. Use Remark 7.2 to construct the family of diagrams (7.3) corresponding to the universal KL-pair $\mathcal{P}$ on GR. The composite map $B\circ A$ is zero if and only if

$$\mathrm{H}^0\left((B\circ A)(1)\right) : \overline{K} = \overline{K}\otimes \mathrm{H}^0(\mathcal{O}) \to \mathrm{H}^0(\overline{L}\otimes\mathcal{O}(2)) = \overline{L}\otimes S_2$$

is zero. But this is the map of vector bundles

$$\phi : \overline{K} \xrightarrow{i} H_{\mathrm{GR}}\otimes S_1 \to H_{\mathrm{GR}}\otimes S_1^*\otimes S_1\otimes S_1 \to \overline{L}\otimes S_1\otimes S_1 \xrightarrow{\overline{L}\otimes \mathrm{mult}} \overline{L}\otimes S_2.$$

The subfunctor of part (1) is therefore represented by the vanishing locus $\mathcal{N}_{\mathcal{B}}$ of this map of vector bundles on $\mathrm{GR}_{\mathcal{B}}$ and so it is closed. By Remark 7.3, $\mathcal{N}_{\mathcal{B}}$ is $\mathrm{GL}(H)$-invariant.

Part (2) follows since the functor represented by $\mathcal{N}_{\mathcal{B}'}$ is the fibre product of the functors represented by $\mathcal{N}_{\mathcal{B}}$ and $\mathcal{B}'$. □

We now follow [LP1] to show that framed semistable Kronecker complexes correspond to points of an open subscheme of $\mathcal{N}$. For this we can specialize to $\mathcal{B} = \operatorname{Spec} k$ and hence work with a fixed AS regular algebra $S \in \underline{\mathrm{AS}}_3$. The conventions concerning $\{r, c_1, \chi\}$ and H from Notation 7.1 will be maintained. Write $\mathrm{GR} = \mathrm{GR}_S$ for the corresponding product $\mathrm{GR}_{\operatorname{Spec} k}$ of Grassmannians associated to S, as in (7.4), and $\mathcal{N} = \mathcal{N}_S \subset \mathrm{GR}$ for the closed subscheme $\mathcal{N}_{\operatorname{Spec} k}$ defined by Lemma 7.4.

The next theorem identifies $\widehat{\mathcal{K}}^{ss}$ as a subscheme of $\mathcal{N}$, for which we need the following construction.

Construction 7.5. Consider a KL-pair (K, L) as in (7.5), where $U = \operatorname{Spec} F$ for some field extension F of k. Given a subspace $H' \subset H$, set $K' = K \cap (H'\otimes S_1)$ and $L' = \mathrm{Im}(H'\otimes S_1^* \to L)$. If (K, L) lies in $\mathcal{N}$, then $K'\otimes\mathcal{O}(-1) \to H'\otimes\mathcal{O} \to L'\otimes\mathcal{O}(1)$ is also a Kronecker complex, and we let r', c_1', χ' denote the rank, first Chern class and Euler characteristic of this complex.

Lemma 7.6. *(1) Let $S \in \underline{\mathrm{AS}}_3$. Then $\widehat{\mathcal{K}}^{ss}$ is $\mathrm{GL}(H)$-equivariantly isomorphic to the subfunctor $\mathcal{N}'$ of $\mathcal{N}$ defined as follows: $\mathcal{N}'(\operatorname{Spec} R)$ consists of those KL-pairs (K, L) such that for each geometric point $\operatorname{Spec} F \to \operatorname{Spec} R$, the KL-pair $(K\otimes F, L\otimes F)$ satisfies the following condition.*

(7.6) *For every $\{0\} \subsetneq H' \subsetneq H_F$, one has $r(c_1'm+\chi') - r'(c_1 m + \chi) \le 0$ as polynomials in m.*

(2) The same result holds for $\widehat{\mathcal{K}}^{s}$ provided one replaces ≤ 0 by < 0 in (7.6).

Proof. (1) Suppose that K and L are vector bundles on $U = \operatorname{Spec} R$ and that

$$\mathbf{K}: \ K\otimes\mathcal{O}_{S_R}(-1) \xrightarrow{A} H\otimes_R \mathcal{O}_{S_R} \xrightarrow{B} L\otimes\mathcal{O}_{S_R}(1)$$

is a family of geometrically semistable Kronecker complexes parametrized by U. By Corollary 6.19, $\mathbf{K}$ is a family of torsion-free monads. The map $\mathrm{H}^0(A(1)) : K \to H \otimes_R S_1$ satisfies $\mathrm{H}^0(A(1)) \otimes F = \mathrm{H}^0(A(1) \otimes F)$ for each geometric point $\operatorname{Spec} F \to \operatorname{Spec} R$. Thus $K \otimes_R F \hookrightarrow H \otimes_R S_1 \otimes_R F$ for each such F. Therefore, by [Ei, Theorem 6.8], $H \otimes_R S_1/K$ is flat, hence a vector bundle on U. Thus K is a subbundle of $H \otimes_R S_1$.

Next, consider the map

$$\phi(B) : H_R \otimes S_1^* \xrightarrow{\mathrm{H}^0(B) \otimes S_1^*} L \otimes S_1 \otimes S_1^* \to L.$$

We claim that $\phi(B)$ is surjective. To see this, note that the composite map

$$\psi : H \to H \otimes S_1^* \otimes S_1 \xrightarrow{\phi(B) \otimes S_1} L \otimes S_1$$

is just $\mathrm{H}^0(B)$. Since $B = \mathrm{H}^0(B) \otimes \mathcal{O}_{S_R}$, this implies that $B = \psi \otimes \mathcal{O}_{S_R}$. Thus, if $\phi(B)$ has image $L' \subsetneq L$, then $\operatorname{Im}(B) \subseteq L' \otimes \mathcal{O}_{S_R}(1)$. Since $(L/L') \otimes S(1)$ is not a bounded S-module it has a nonzero image in qgr-S_R, contradicting the surjectivity of B. Thus $\phi(B)$ is surjective and L is a quotient bundle of $H \otimes_R S_1$.

Thus $\mathbf{K} \mapsto (K, L)$ gives a map $\alpha : \widehat{\mathcal{K}}^{ss} \to \mathrm{GR}$. It is easily checked that the construction of Remark 7.2 sends this pair (K, L) back to the original Kronecker complex $\mathbf{K}$. Thus, α is injective and (by Lemma 7.4) α factors through $\mathcal{N}$. Since a geometrically semistable monad certainly satisfies (7.6) at each geometric point, the image of α is contained in $\mathcal{N}'$. The equivariance of this embedding of functors follows from the last sentence of Remark 7.3.

It remains to show that α surjects onto $\mathcal{N}'$. Suppose that $(K, L) \in \mathcal{N}(\operatorname{Spec} R)$ is a KL-pair for which $(K \otimes F, L \otimes F)$ satisfies (7.6) for every geometric point $\operatorname{Spec} F \to \operatorname{Spec} R$. Let $\mathbf{K}$ be the Kronecker complex determined by (K, L). Suppose that

$$\mathbf{K}' : \quad K' \otimes \mathcal{O}(-1) \to H' \otimes \mathcal{O} \to L' \otimes \mathcal{O}(1)$$

is a maximal subcomplex of $\mathbf{K}_F$ for some geometric point $\operatorname{Spec} F$, as defined in the proof of Proposition 6.20. Then, as in [LP1, Lemme 2.1], it is straightforward to check that $\mathbf{K}'$ is the subcomplex associated to H' by Construction 7.5. Since $\mathbf{K}_F$ satisfies (7.6), it follows that $\operatorname{rk}(\mathbf{K}_F) p_{\mathbf{K}'} - \operatorname{rk}(\mathbf{K}') p_{\mathbf{K}_F} \leq 0$, implying that $\mathbf{K}_F$ is semistable. Consequently $\mathbf{K}$ is geometrically semistable.

(2) The proof for stable complexes is essentially the same. □

We next prove that $\mathcal{N}'$ is exactly the semistable locus of $\mathcal{N}$ in the GIT sense for the action of $\mathrm{SL}(H)$. We proceed as follows. For each fixed r, c_1 and χ there are only finitely many possible values of r', c_1' and χ' that can occur as invariants associated to a subspace $H' \subset H$. Thus, for any integer $m = m(r, c_1, \chi) \gg 0$ and for all possible values of r', c_1' and χ' associated to subspaces of H, the inequality in (7.6) is satisfied if and only if it is satisfied for this fixed value $m(r, c_1, \chi)$ of m.

The variety GR has natural ample line bundles $\mathcal{O}(k, \ell)$ obtained by pulling back $\mathcal{O}(k)$, $\mathcal{O}(\ell)$ from projective spaces under the Plücker embeddings of the two factors of GR (see [HL, Example 2.2.2]) and these bundles are equivariant for the diagonal $\mathrm{GL}(H)$-action. The open subset of $\mathcal{N}$ consisting of (semi)stable points (see [Se, Definition 2]) for the action of $\mathrm{SL}(H)$ under the linearization $\mathcal{O}(k, \ell)$ of GR will be written $\mathcal{N}^s$, respectively $\mathcal{N}^{ss}$.

One now has a result analogous to [LP1, Théorème 3.1].

Proposition 7.7. *Let $S \in \underline{\mathrm{AS}}_3$, pick integers $\{r \geq 1, c_1, \chi\}$ and choose $m = m(r, c_1, \chi) \gg 0$ as above. Set $k = (2m+3)(c_1+r)-n$ and $\ell = -(2m+3)(c_1-r)+n$. Then $\widehat{\mathcal{K}}^s = \mathcal{N}^s$ and $\widehat{\mathcal{K}}^{ss} = \mathcal{N}^{ss}$ under the linearization $\mathcal{O}(k, \ell)$ of* GR.

Proof. Let F be an algebraically closed field. By [LP1, Lemme 3.3], a pair $(K, L) \in \mathrm{GR}(\operatorname{Spec} F)$ is (semi)stable for the $\mathrm{SL}(H)$-action if and only if

$$k[n \dim K' - n' \dim K] - \ell[n \dim L' - n' \dim L] < 0 \text{ (respectively } \leq 0) \tag{7.7}$$

for every proper nonzero $H' \subset H_F$ of dimension n' (where K' and L' are defined by Construction 7.5).

Now consider the point (K, L) associated to a Kronecker complex $\mathbf{K}$. Use (7.2) to rewrite the left hand side of (7.7) in terms of $r, \ldots, d_1'$. After simplification, this gives:

$$k\,[n \dim K' - n' \dim K] - \ell\,[n \dim L' - n' \dim L] = Z(H'), \tag{7.8}$$

where $Z(H') = 2n\,[r(c_1' m + \chi') - r'(c_1 m + \chi)]$. By Lemma 7.6 and the choice of m, $Z(H') < 0$ (respectively $Z(H') \leq 0$) for every $H' \subset H$ if and only if the Kronecker complex $\mathbf{K}$ associated to (K, L) is (semi)stable. By the first paragraph of the proof, (7.8) implies that the (semi)stability of the Kronecker complex $\mathbf{K}$ is equivalent to the (semi)stability of the associated point (K, L) of the Grassmannian. □

Corollary 7.8. *Let $S \in \underline{\mathrm{AS}}_3$ and pick integers $\{r \geq 1, c_1, \chi\}$. Then $\mathcal{N}^s$ and $\mathcal{N}^{ss}$ are fine moduli spaces for isomorphism classes of geometrically (semi)stable framed Kronecker complexes with invariants $\{r, c_1, \chi\}$.* □

7.2. **Existence of Moduli Spaces.** Corollary 7.8 does not directly give fine moduli spaces for isomorphism classes of S-modules, since Kronecker complexes cannot be canonically framed in families. (In fact, as in the commutative setting [HL], one can only hope to get a fine moduli space for equivalence classes of modules, as defined in Definition 6.18.) However, using the results of Sections 5 and 6 it is not hard to produce the relevant coarse and fine moduli spaces.

Before stating the result, we show that, although the last subsection required the invariants $\{r, c_1, \chi\}$ to satisfy $-r < c_1 \leq 0$, the results proved there can be applied to S-modules with any prespecified invariants. We emphasize, however, that the spaces $\mathcal{N}^s$, etc, have only been defined for invariants satisfying the hypotheses of Notation 7.1 and we do not wish to define them more generally.

Remark 7.9. Let $S \in \underline{\mathrm{AS}}_3$ and pick a commutative noetherian k-algebra R. Write $\mathcal{C}(r, c_1, \chi)$ for the category of R-flat families of geometrically semistable (or stable) torsion-free modules $\mathcal{M} \in \text{qgr-}S_R$ with invariants $\{r, c_1, \chi\}$. If $\mathcal{M} \in \mathcal{C}(r, c_1, \chi)$, it follows from Lemma 3.7(2) and Corollary 6.2 that $c_1(\mathcal{M}(1)) = c_1 + r$ and $\chi((\mathcal{M}(1)) = \chi + 2r + c_1$. Thus, by induction, there exist unique invariants $\{r, c_1', \chi'\}$ such that $-r < c_1' \leq 0$ and $\mathcal{C}(r, c_1, \chi) \simeq \mathcal{C}(r, c_1', \chi')$.

The next result proves Theorem 1.6 from the introduction.

Theorem 7.10. *Let $S \in \underline{\mathrm{AS}}_3$ and fix integers r, c_1, χ, with $r \geq 1$.*

(1) *There exists a projective coarse moduli space $\mathcal{M}_S^{ss}(r, c_1, \chi)$ for equivalence classes of geometrically semistable torsion-free modules in* qgr-S *of rank r, first Chern class c_1 and Euler characteristic χ.*
(2) *$\mathcal{M}_S^{ss}(r, c_1, \chi)$ contains an open subscheme $\mathcal{M}_S^{s}(r, c_1, \chi)$ that is a coarse moduli space for the geometrically stable modules.*

(3) *If* $-r < c_1 \leq 0$ *then* $\mathcal{M}_S^{ss}(r, c_1, \chi) = \mathcal{N}^{ss} /\!\!/ \mathrm{PGL}(H)$ *and* $\mathcal{M}_S^{s}(r, c_1, \chi) = \mathcal{N}^{s} /\!\!/ \mathrm{PGL}(H)$.

Proof. By Remark 7.9 we may assume in parts (1) and (2) that $-r < c_1 \leq 0$ and hence that our modules are normalized. By Proposition 6.20, the moduli functor for equivalence classes of geometrically semistable torsion-free modules in qgr-S with invariants $\{r, c_1, \chi\}$ is now isomorphic to $\mathcal{K}^{ss}$, the functor for the corresponding Kronecker complexes. So, it suffices to prove the theorem for $\mathcal{K}^{ss}$.

The group functor $\mathrm{PGL}(H)$ acts on $\widehat{\mathcal{K}}^{ss}$, and there is a (forgetful) map of functors $\widehat{\mathcal{K}}^{ss}/\mathrm{PGL}(H) \to \mathcal{K}^{ss}$. Although this map is not an isomorphism of functors, we claim that it is a *étale local isomorphism* in the sense that it induces an isomorphism of the sheafifications for the étale topology of affine k-schemes [Si, p.60]. To see this, observe that the map of functors is injective, so it suffices to prove that it is étale locally surjective. Let

$$\mathbf{K} : \mathcal{O}_{S_R}(-1) \otimes_R V_{-1} \to \mathcal{O}_{S_R} \otimes V_0 \to \mathcal{O}_{S_R}(1) \otimes V_1$$

be an R-flat family of geometrically semistable Kronecker complexes, as in (6.18). After an étale base change $R \to R'$, the R-module V_0 becomes trivial and so $\mathbf{K} \otimes_R R'$ is in the image of $\widehat{\mathcal{K}}^{ss}(R')$, proving the claim.

By [Si, p.60], this implies that any scheme Z that corepresents $\widehat{\mathcal{K}}^{ss}/\mathrm{PGL}(H)$ also corepresents $\mathcal{K}^{ss}$. But Proposition 7.7 implies that $\mathcal{N}^{ss} = \widehat{\mathcal{K}}^{ss}$, while [HL, Theorem 4.2.10] implies that the GIT quotient $Z = \mathcal{N}^{ss} /\!\!/ \mathrm{PGL}(H)$ does corepresent $\widehat{\mathcal{K}}^{ss}/\mathrm{PGL}(H)$. So Z corepresents $\mathcal{K}^{ss}$ and gives the desired moduli space. □

Remark 7.11. Pick integers $\{r, c_1, \chi\}$ with $-r < c_1 \leq 0$. For future reference, note that, by Corollary 7.8 and third paragraph of the proof of Theorem 7.10, the stack-theoretic quotient $[\mathcal{N}^s/GL(H)]$ equals the moduli stack of geometrically stable Kronecker complexes with invariants $\{r, c_1, \chi\}$.

Corollary 7.12. *Let* $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}_3'$ *and pick integers* $\{r \geq 1, c_1, \chi\}$. *Then there is an open subscheme of* $\mathcal{M}_S^s(r, c_1, \chi)$ *parametrizing modules whose restriction to* E *is locally free.*

Proof. Suppose that one is given a flat family of coherent sheaves on E parametrized by a scheme X. Then the vector bundles in that family are parametrized by an open subset of X. Thus, the corollary follows from the next lemma. □

Lemma 7.13. *Let* $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}_3'$. *Let* $\mathcal{M}$ *be an* R*-flat family of torsion-free objects in* qgr-S_R. *Then* $\mathcal{M}|_E$ *is* R*-flat.*

Proof. By [AZ2, Lemma E5.3], $M = \Gamma^*(\mathcal{M})_{\geq n}$ is a flat R-module for any $n \gg 0$ and so it suffices to show that M/Mg is R-flat. By [AZ2, Lemma C1.12] it is enough to show that $\mathrm{Tor}_1^R(M/Mg, R/P) = 0$ for every prime ideal $P \subset R$. By [AZ2, Proposition C1.9] we may assume that $R = (R, P)$ is local.

Write S for S_R. By hypothesis, $\mathcal{M}/\mathcal{M}P$ is a torsion-free module in qgr-$S_{R/P}$ and so, possibly after increasing n, M/MP is a torsion-free module in gr-$S_{R/P}$. Thus g acts without torsion. Equivalently, $Mg \cap MP = MPg$ and so the natural homomorphism $MP \otimes_S S/gS \to M \otimes_S S/gS$ is injective. As M_R is flat, $MP \cong M \otimes_R P$ and so the natural map

$$\theta : (M \otimes_R P) \otimes_S S/gS \to M \otimes_S S/gS \cong M/Mg$$

is injective. Since the actions of S and R on M commute,

$$(M \otimes_R P) \otimes_S S/gS \cong (M \otimes_S S/gS) \otimes_R P \cong M/Mg \otimes_R P.$$

Thus θ is just the natural map $M/Mg \otimes_R P \to M/Mg \otimes_R R$. In particular, $\mathrm{Tor}_1^R(\mathcal{M}|_E, R/P) = \pi\,\mathrm{Ker}(\theta) = 0$. □

In many cases the stable locus of the moduli space of Theorem 7.10 is a fine moduli space and the next proposition gives one such example. The general technique for producing fine moduli spaces is nicely explained in [HL, Section 4.6], but we will give a detailed proof of this result for the benefit of the reader who is unfamiliar with the main ideas. We first need a variant of the standard fact that stable objects have trivial endomorphism rings when the ground field k is algebraically closed.

Lemma 7.14. *Suppose that $\mathcal{F}$ is a geometrically stable torsion-free object in* qgr-S_F *for a field F. Then the natural map $F \to \mathrm{Hom}_{\text{qgr-}S_F}(\mathcal{F}, \mathcal{F})$ is an isomorphism.*

Proof. Suppose that $F \to F'$ is a field extension and that $\phi : \mathcal{F} \otimes F' \to \mathcal{F} \otimes F'$ is a nonzero endomorphism. By the observation before Lemma 6.3, $\mathcal{F} \otimes F'$ is stable. Since $\mathrm{Im}(\phi)$ is both a subobject and a quotient object of $\mathcal{F} \otimes F'$, this implies that $\mathrm{rk}(\mathcal{F} \otimes F')p_{\mathrm{Im}(\phi)} - \mathrm{rk}(\mathrm{Im}(\phi))p_{\mathcal{F}\otimes F'}$ is both positive and negative. This contradicts stability unless $\mathrm{Im}(\phi) = \mathcal{F} \otimes F'$. Thus, ϕ is an automorphism and $\mathrm{End}(\mathcal{F} \otimes F')$ is a division ring containing F'.

By [AZ1, Theorem 7.4], $\mathrm{End}(\mathcal{F} \otimes F')$ is a finite-dimensional F'-vector space. Thus $F' = \mathrm{End}(\mathcal{F} \otimes F')$ if F' is algebraically closed. In particular, if $\overline{F}$ is the algebraic closure of F, then $\mathrm{End}(\mathcal{F}) \otimes \overline{F} \subseteq \mathrm{End}(\mathcal{F} \otimes \overline{F}) = \overline{F}$ and so the natural map $F \to \mathrm{End}(\mathcal{F})$ must be an isomorphism. □

Proposition 7.15. *Let $S \in \underline{\mathrm{AS}}_3(k)$, where* char $k = 0$*. Fix integers $\{r, c_1, \dots, d_1\}$ by Notation* (7.1)*. If* $\mathrm{GCD}(d_{-1}, n, d_1) = 1$*, then $\mathcal{M}_S^s(r, c_1, \chi)$ is a fine moduli space whenever it is nonempty.*

Proof. Theorem 7.10 defines a map of functors $a : \mathcal{K}^s \to \mathcal{M}^s$. We think of both functors as presheaves on the category of (noetherian affine) k-schemes in the étale topology. Then $\mathcal{M}^s$ is a sheaf since it is represented by a scheme. We will construct a map $b : \mathcal{M}^s \to \mathcal{K}^s$ of functors and show that

(i) $b \circ a = 1_{\mathcal{K}^s}$
(ii) a is étale-locally surjective, in the sense that for any affine noetherian k-scheme U and map $x \in \mathrm{Hom}(U, \mathcal{M}^s)$ there is an étale cover $V \to U$ and object $\tilde{x}$ in $\mathcal{K}^s(V)$ such that the image of $\tilde{x}$ in $\mathrm{Hom}(V, \mathcal{M}^s)$ is $x|_V$.

The isomorphism of $\mathcal{K}^s$ and $\mathcal{M}^s$ is then a special case of the following general fact (proved by a simple diagram chase): suppose that F is a presheaf of sets and G is a sheaf of sets. Suppose further that $a : F \to G$ and $b : G \to F$ are natural transformations of these presheaves such that $b \circ a = 1_F$ and such that a is étale-locally surjective. Then $F = G$.

We first prove (ii). Let $x \in \mathrm{Hom}(U, \mathcal{M}^s)$. As in the proof of [HL, Corollary 4.2.13], Luna's étale slice theorem together with Lemma 7.14 implies that the projection $\mathcal{N}^s \to \mathcal{M}^s$ is a principal $\mathrm{PGL}(H)$-bundle. It follows that there exists an étale cover $V \to U$ such that the fibre product $\mathcal{N}^s \times_{\mathcal{M}^s} V$ is a $\mathrm{PGL}(H)$-bundle with a section. Hence $x|_V$ is in the image of the map $\mathcal{N}^s(V) \to \mathcal{M}^s(V)$. But this map factors through $\mathcal{K}^s(V)$, so there is an element $\tilde{x} \in \mathcal{K}^s(V)$ lifting $x|_V$.

We next construct the map b. By Yoneda's Lemma, in order to produce b it is enough to produce a family $\mathcal{U}$ of geometrically stable Kronecker complexes in qgr-$S\otimes\mathcal{O}_{\mathcal{M}^s}$. The universal sub and quotient modules K and L on GR are $\mathrm{GL}(H)$-equivariant and the centre $\mathbf{G}_m$ acts with weight 1 in their fibres. Pick $a,b,c\in\mathbb{Z}$ satisfying $ad_{-1}+bd_1+cn=1$. Then

$$M=(\det K)^{\otimes a}\otimes(\det L)^{\otimes b}\otimes(\det H)^{\otimes c}$$

is a $\mathrm{GL}(H)$-equivariant line bundle on GR and $\mathbf{G}_m$ acts with weight 1 in its fibres (see Remark 7.3). Hence

$$\mathcal{P}'=(K\otimes M^*\to S_1\otimes H_{\mathcal{N}^s}\otimes M^*,\ S_1^*\otimes H_{\mathcal{N}^s}\otimes M^*\to L\otimes M^*)$$

forms a pair of $\mathrm{GL}(H)$-equivariant maps on $\mathcal{N}^s$ with $\mathbf{G}_m$ acting trivially on the sheaves.

These sheaves and maps $\mathcal{P}'$ are therefore $\mathrm{PGL}(H)$-equivariant. Since $\mathcal{N}^s\to\mathcal{M}^s$ is a principal $\mathrm{PGL}(H)$-bundle, $\mathcal{P}'$ descends to $\mathcal{M}^s$. Now Remark 7.2 associates to $\mathcal{P}'$ a diagram $\mathcal{U}$ in qgr-$S\otimes\mathcal{O}_{\mathcal{M}^s}$. The pullback of $\mathcal{U}$ to $\mathcal{N}^s$ is the Kronecker complex associated to the universal KL-pair on $\mathcal{N}^s$ and $\mathcal{N}^s\to\mathcal{M}^s$ is faithfully flat. Thus [AZ2, Lemma C1.1] implies that $\mathcal{U}$ is a complex. Furthermore, any map $\operatorname{Spec}F\to\mathcal{M}^s$ for a field F lifts to $\operatorname{Spec}F\to\mathcal{N}^s$. Thus $\mathcal{U}$ is a geometrically stable Kronecker complex since its pullback to $\mathcal{N}^s$ is. This defines the map b.

Finally, we will prove that $\mathcal{U}$ is *weakly universal* in the following sense: any family of geometrically stable Kronecker complexes in qgr-$S\otimes\mathcal{O}_U$ for a noetherian affine k-scheme U is equivalent to the pullback of $\mathcal{U}$ under the induced composite $U\to\mathcal{K}^s\to\mathcal{M}^s$. Yoneda's Lemma will then imply that $b\circ a=1_{\mathcal{K}^s}$.

Suppose that U is an affine noetherian k-scheme with a family $\mathbf{K}$ of geometrically stable Kronecker complexes (6.18). Let $\mathrm{Fr}(V_0)\xrightarrow{p}U$ denote the bundle whose fibre over $u\in U$ consists of vector space isomorphisms of the fibre $(V_0)_u$ with H. This is a principal $\mathrm{GL}(H)$-bundle over U and the pullback of V_0 to $\mathrm{Fr}(\mathcal{H})$ comes equipped with a canonical isomorphism with $H\otimes\mathcal{O}_{\mathrm{Fr}(V_0)}$. Thus, $\mathbf{K}$ pulls back to a framed Kronecker complex $p^*\mathbf{K}$ on $\mathrm{Fr}(V_0)$. This pullback determines a $\mathrm{GL}(H)$-equivariant map $\mathrm{Fr}(V_0)\to\mathcal{N}^s$ by Corollary 7.8 and so we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Fr}(V_0) \arrow[r, "f_1"] \arrow[d, "p"] & \mathcal{N}^s \arrow[d, "q"] \\
U \arrow[r, "f_2"] & \mathcal{M}^s
\end{tikzcd}$$

where f_2 is the canonical composite $U\to\mathcal{K}^s\to\mathcal{M}^s$. By construction, then, we have $\mathrm{GL}(H)$-equivariant isomorphisms

$$p^*\mathbf{K}=f_1^*(q^*\mathcal{U}\otimes M)=p^*f_2^*\mathcal{U}\otimes f_1^*M.$$

Since f_1^*M is a $\mathrm{GL}(H)$-equivariant line bundle on $\mathrm{Fr}(V_0)$, it descends to a line bundle $\overline{M}$ on U. So there is a $\mathrm{GL}(H)$-equivariant isomorphism $p^*(\mathbf{K}\otimes\overline{M}^*)\cong p^*f_2^*\mathcal{U}$. This isomorphism then descends to show that $\mathbf{K}\otimes\overline{M}\cong f_2^*\mathcal{U}$. In other words, $\mathbf{K}$ is equivalent to the pullback of $\mathcal{U}$ from $\mathcal{M}^s$ by the natural map. This shows that $b\circ a=1_{\mathcal{K}^s}$ and completes the proof of the proposition. $\square$

An analogue of Proposition 7.15 holds for all values of the invariants $\{r\geq 1,c_1,\chi\}$ but is awkward to state precisely since the condition on GCD's must be assumed

for the normalization of the given modules. The next result gives two special cases where it is easy to unravel this condition.

Corollary 7.16. *Let* $S \in \underline{\mathrm{AS}}_3(k)$ *with* char $k = 0$ *and fix integers* $\{r \geq 1, c_1, \chi\}$. *Assume that* either (1) $c_1 = 0$ *and* $\mathrm{GCD}(r, \chi) = 1$ or (2) $\mathrm{GCD}(r, c_1) = 1$.

Then $\mathcal{M}^s_S(r, c_1, \chi)$ *is a fine moduli space whenever it is nonempty.*

Proof. (1) Use (7.2) to see that this case is already covered by Proposition 7.15.

(2) By Remark 7.9, the hypothesis $\mathrm{GCD}(r, c_1) = 1$ is unchanged when one shifts a module and so we can assume that $-r < c_1 \leq 0$. Now, (7.2) implies that $\mathrm{GCD}(d_{-1}, n, d_1) = 1$ and so the result follows from Proposition 7.15. □

The next result gives a simple application of this observation.

Corollary 7.17. *Let* $S \in \underline{\mathrm{AS}}_3(k)$, *where* $\mathrm{char}\, k = 0$. *For* $n \geq 1$, $\mathcal{M}^{ss}(1, 0, 1-n) = \mathcal{M}^s(1, 0, 1-n)$ *is a nonempty, projective, fine moduli space for equivalence classes of rank one torsion-free modules* $\mathcal{M}$ *in* qgr-S *with* $c_1(\mathcal{M}) = 0$ *and* $\chi(\mathcal{M}) = 1 - n$.

If qgr-$S \simeq \mathrm{coh}(\mathbf{P}^2)$, *then* $\mathcal{M}^{ss}(1, 0, 1-n)$ *is isomorphic to the Hilbert scheme* $(\mathbf{P}^2)^{[n]}$ *of* n *points on* $\mathbf{P}^2$.

Proof. By Corollary 6.6(1), $\mathcal{M}^{ss}(1, 0, 1-n) = \mathcal{M}^s(1, 0, 1-n)$. We next check that there exist torsion-free modules with the specified invariants. Given a point module $P \in$ gr-S, [ATV2, Proposition 6.7(i)] implies that it has a resolution

$$0 \to \mathcal{O}_S(-2) \to \mathcal{O}_S(-1)^2 \to \mathcal{O}_S \to P \to 0,$$

from which it follows that $c_1(P) = 0$ but $\chi(P) = 1$. Thus, if one takes n nonisomorphic point modules $P_i \cong S/I_i$, then $M = \bigcap I_i$ is a torsion-free module with $c_1(M) = 0$ and $\chi(M) = 1 - n$, as required. Now, Theorem 7.10 and Corollary 7.15 combine to prove that $\mathcal{M}^{ss}(1, 0, 1-n)$ is a projective fine moduli space. The final assertion is standard—see for example [HL, Example 4.3.6]. □

Suppose one has a family $S_{\mathcal{B}}(E, \mathcal{L}, \sigma)$ of algebras in $\underline{\mathrm{AS}}_3$ that give a deformation of the polynomial ring $k[x, y, z]$. Then, as we will show in the next section, one can view Corollary 7.17 as showing that $\mathcal{M}^{ss}(1, 0, 1-n)$ is a deformation of $(\mathbf{P}^2)^{[n]}$. Since we proved that $\mathcal{M}^{ss}(1, 0, 1-n)$ is nonempty by finding the module corresponding to n points on E, this may not seem so surprising. However this analogy does not work for the subset $(\mathbf{P}_S \smallsetminus E)^{[n]} \subset \mathcal{M}^{ss}(1, 0, 1-n)$ that deforms $(\mathbf{P}^2 \smallsetminus E)^{[n]}$. The reason is that, when $|\sigma| = \infty$, the only "points" in qgr-S are those that lie on E and so, necessarily, the modules parametrized by $(\mathbf{P}_S \smallsetminus E)^{[n]}$ have to be more subtle. In fact they are line bundles—see Theorem 8.11(3). These modules correspond in turn to the projective right ideals of the ring $A(S)$ that we considered in Section 3.

8. Moduli Spaces: Properties

In this section we study the more detailed properties of the moduli spaces constructed in the last section. We prove that, if nonempty, they are smooth and behave well in families (see Theorem 8.1). Moreover, in many cases they are irreducible and hence connected (see Proposition 8.6). In rank one, we are able to give a much more detailed picture of $\mathcal{M}^{ss}(1, 0, 1-n)$ and its open subvariety $(\mathbf{P}_S \smallsetminus E)^{[n]}$, thereby justifying the comments from the end of the last section.

8.1. **Basic Properties.** We first consider a family of AS regular algebras $S_{\mathcal{B}}$ parametrized by a k-scheme $\mathcal{B}$, where char $k = 0$, since we wish to study the global structure of our moduli spaces. Pick invariants $\{r, c_1, \chi\}$ as in Notation 7.1. Then Lemma 7.4 and general properties of GIT quotients in the relative setting [Se, Remark 8] yield a subscheme $\mathcal{N}^{ss}_{\mathcal{B}}$ of $\mathrm{GR}_{\mathcal{B}}$ such that $\mathcal{N}^{ss}_{\mathcal{B}} /\!\!/ \mathrm{PGL}(H)$ is a universal categorical quotient in the sense that for any $\mathcal{B}$-scheme $\mathcal{B}' \to \mathcal{B}$, the fibre product $(\mathcal{N}^{ss}_{\mathcal{B}} /\!\!/ \mathrm{PGL}(H)) \times_{\mathcal{B}} \mathcal{B}'$ corepresents the quotient functor $\mathcal{N}^{ss}_{\mathcal{B}'} / \mathrm{PGL}(H)$. In particular, the fibre of this quotient over $b \in \mathcal{B}$ is $\mathcal{M}^{ss}_{S_b}(r, c_1, \chi)$ and the same holds for the stable locus. We write $\mathcal{M}^{s}_{\mathcal{B}}(r, c_1, \chi) = \mathcal{N}^{s}_{\mathcal{B}} /\!\!/ \mathrm{PGL}(H)$. Applying Luna's étale slice theorem as in [HL, Corollary 4.2.13], and using Lemma 7.14, shows that the projection $\mathcal{N}^{s}_{\mathcal{B}} \to \mathcal{M}^{s}_{\mathcal{B}}(r, c_1, \chi)$ is a principal $\mathrm{PGL}(H)$-bundle. When $\{r \geq 1, c_1, \chi\}$ are arbitrary integers, use Remark 7.9 to choose the unique integers $\{r, c_1', \chi'\}$ with $-r < c_1' \leq 0$ and $\mathcal{C}(r, c_1, \chi) \simeq \mathcal{C}(r, c_1', \chi')$. Then set $\mathcal{M}^{ss}_{\mathcal{B}}(r, c_1, \chi) = \mathcal{M}^{ss}_{\mathcal{B}}(r, c_1', \chi')$ and $\mathcal{M}^{s}_{\mathcal{B}}(r, c_1, \chi) = \mathcal{M}^{s}_{\mathcal{B}}(r, c_1', \chi')$.

We are now ready to prove Theorem 1.7 from the introduction.

Theorem 8.1. *Suppose that* char $k = 0$ *and that S is a flat family of algebras in* $\underline{\mathrm{AS}}_3$ *parametrized by a k-scheme $\mathcal{B}$. If $\{r \geq 1, c_1, \chi\} \subset \mathbb{Z}$, then*

(1) *$\mathcal{M}^{s}_{\mathcal{B}}(r, c_1, \chi)$ is smooth over $\mathcal{B}$.*
(2) *If $p \in \mathcal{B}$ then $\mathcal{M}^{s}_{\mathcal{B}}(r, c_1, \chi) \otimes_{\mathcal{B}} k(p) = \mathcal{M}^{s}_{S_b}(r, c_1, \chi)$, and similarly for $\mathcal{M}^{ss}_{\mathcal{B}}(r, c_1, \chi)$.*

Proof. Part (2) is a general property of universal quotients [Se], so only part (1) needs proof. We may assume that the invariants satisfy $-r < c_1 \leq 0$.

Since the projection $\mathcal{N}^{s}_{\mathcal{B}} \to \mathcal{M}^{s}_{\mathcal{B}}(r, c_1, \chi)$ is a principal bundle, [EGA, Proposition IV.17.7.7] implies that it is enough to prove that $\mathcal{N}^{s}_{\mathcal{B}}$ is smooth over $\mathcal{B}$. To prove this we will use the local criterion for smoothness. Let R' be a local commutative k-algebra with a noetherian factor ring $R = R'/I$, where $I^2 = 0$. Suppose that

$$\mathbf{K}_R: \quad \mathcal{O}_{S_R}(-1) \otimes V_{-1} \xrightarrow{d_R^1} \mathcal{O}_{S_R} \otimes V_0 \xrightarrow{d_R^2} \mathcal{O}_{S_R}(1) \otimes V_1$$

is an R-flat family of stable monads. Because $R = (R, \mathfrak{m})$ is local, we may assume that the V_i are k-vector spaces and the tensor products are over k.

Suppose that we can lift $\mathbf{K}$ to a complex

$$\mathbf{K}_{R'}: \quad \mathcal{O}_{S_{R'}}(-1) \otimes V_{-1} \xrightarrow{d_{R'}^1} \mathcal{O}_{S_{R'}} \otimes V_0 \xrightarrow{d_{R'}^2} \mathcal{O}_{S_{R'}}(1) \otimes V_1 \tag{8.1}$$

satisfying $\mathbf{K}_{R'} \otimes_{R'} R \cong \mathbf{K}_R$. Then $\mathbf{K}_{R'} \otimes_{R'} I \cong \mathbf{K}_R \otimes_R I$ and so Lemma 5.9 implies that $\mathrm{H}^n(\mathbf{K}_{R'} \otimes_{R'} I) = \mathrm{H}^n(\mathbf{K}_R) \otimes_R I$. From the long exact cohomology sequence associated to the exact sequence of complexes

$$0 \to \mathbf{K}_{R'} \otimes I \to \mathbf{K}_{R'} \to \mathbf{K}_R \to 0,$$

and the vanishing of $\mathrm{H}^n(\mathbf{K}_R)$ for $n \neq 0$, we find that $\mathbf{K}_{R'}$ is automatically a monad. Moreover, from the local criterion for flatness [AZ2, Proposition C7.1], the cohomology of $\mathbf{K}_{R'}$ is flat over R'. Thus, $\mathcal{N}^s$ satisfies the lifting property and so $\mathcal{M}^{s}_{\mathcal{B}}(r, c_1, \chi)$ is smooth.

So it remains to show that we can always lift $\mathbf{K}$ to $\mathbf{K}'$. Choose any lift of $\mathbf{K}_R$ to a diagram $\mathbf{K}_{R'}$ of the form (8.1). We need to adjust the differential $d_{R'}$ to convert $\mathbf{K}_{R'}$ into a complex. Note that $d_{R'} \in \mathrm{Hom}^1(\mathbf{K}_{R'}, \mathbf{K}_{R'})$, where

$$\mathrm{Hom}^j(\mathbf{K}_{R'}, \mathbf{K}_{R'}) = \bigoplus_i \mathrm{Hom}(\mathbf{K}^i_{R'}, \mathbf{K}^{i+j}_{R'}),$$

and $d_{R'} \circ d_{R'} \in \mathrm{Hom}^2(\mathbf{K}_{R'}, \mathbf{K}_{R'})$. The only nonzero term in this product is $d^2_{R'} \circ d^1_{R'}$. Since $d_{R'}$ reduces to d_R mod I, we find that

$$d_{R'} \circ d_{R'} \in \mathrm{Hom}^2(\mathbf{K}_{R'}, \mathbf{K}_{R'} \otimes I) \cong \mathrm{Hom}^2(\mathbf{K}_R, \mathbf{K}_R \otimes_R I).$$

As in the proof of Lemma 5.11, equip $\mathrm{Hom}^\bullet(\mathbf{K}_R, \mathbf{K}_R \otimes I)$ with the differential $\delta(f) = d_{R'} \circ f - (-1)^{\deg(f)} f \circ d_{R'}$. Then $\delta(d_{R'} \circ d_{R'}) = 0$ and so $d_{R'} \circ d_{R'}$ defines a class in $\mathrm{H}^2(\mathrm{Hom}^\bullet(\mathbf{K}_R, \mathbf{K}_R \otimes I))$.

Let $\mathcal{E} = \mathrm{H}^0(\mathbf{K}_R)$ and set $\tilde{k} = R/\mathfrak{m}$. Since $\mathcal{E}$ is geometrically stable, Lemma 7.14 implies that

$$\mathrm{Ext}^2_{\mathrm{Qgr}\text{-}S_{\tilde{k}}}(\mathcal{E} \otimes \tilde{k}, \mathcal{E} \otimes \tilde{k}) = \mathrm{Ext}^0_{\mathrm{Qgr}\text{-}S_{\tilde{k}}}(\mathcal{E} \otimes \tilde{k}, (\mathcal{E} \otimes \tilde{k})(-3))^* = 0.$$

By Theorem 4.3(1,4) this implies that $\mathrm{Ext}^2_{\mathrm{Qgr}\text{-}S_R}(\mathcal{E}, \mathcal{E}) \otimes \tilde{k} = 0$ and then Nakayama's Lemma gives $\mathrm{Ext}^2_{\mathrm{Qgr}\text{-}S_R}(\mathcal{E}, \mathcal{E}) = 0$. Since Qgr-$S$ has homological dimension two, $\mathrm{Ext}^3_{\mathrm{Qgr}\text{-}S_R}(\mathcal{E}, -) = 0$. Writing I as a factor of a free R-module, this implies that $\mathrm{Ext}^2_{\mathrm{Qgr}\text{-}S_R}(\mathcal{E}, \mathcal{E} \otimes I) = 0$. Consequently, by Lemma 5.11, there exists $\phi \in \mathrm{Hom}^1(\mathbf{K}_R, \mathbf{K}_R \otimes I)$ such that $\delta(\phi) = d_{R'} \circ d_{R'}$. Finally, if $D_{R'} = (d_{R'} - \phi) \in \mathrm{Hom}^1(\mathbf{K}_{R'}, \mathbf{K}_{R'})$, then $D_{R'} = d_R$ mod I and

$$D_{R'} \circ D_{R'} = (d_{R'} - \phi) \circ (d_{R'} - \phi) = d_{R'} \circ d_{R'} - \delta(\phi) = 0.$$

Thus $D_{R'}$ determines the desired complex $\mathbf{K}_{R'}$. □

Remark 8.2. The proof of Theorem 8.1 also proves its stack-theoretic analogue. In other words, if one keeps the hypotheses of the proposition and assumes that $-r < c_1 \leq 0$, then the stack-theoretic quotient $[\mathcal{N}^s_{\mathcal{B}}/GL(H)]$ is smooth over $\mathcal{B}$.

Taking $\mathcal{B} = k$ in the theorem gives:

Corollary 8.3. *If S is in $\underline{\mathrm{AS}}_3(k)$ where* $\mathrm{char}\, k = 0$*, then $\mathcal{M}^s_S(r, c_1, \chi)$ is a smooth quasi-projective k-scheme for any integers $\{r \geq 1, c_1, \chi\}$.* □

Corollary 8.4. *Let $S \in \underline{\mathrm{AS}}_3(k)$ where* $\mathrm{char}\, k = 0$ *and pick integers $\{r \geq 1, c_1, \chi\}$. If $\mathcal{M}^s_S(r, c_1, \chi)$ is nonempty, then* $\dim \mathcal{M}^s_S(r, c_1, \chi) = r^2 + 3rc_1 + c_1^2 - 2r\chi + 1$.

Proof. Pick a geometrically stable module $\mathcal{E}$ with invariants $\{r, c_1, \chi\}$. As in [HL, Theorem 4.5.1 and Corollary 4.5.2], the tangent space to $\mathcal{M}^s_S(r, c_1, \chi)$ at the (necessarily smooth) point $[\mathcal{E}]$ is $T_{[\mathcal{E}]}\mathcal{M}^s_S = \mathrm{Ext}^1_{\mathrm{qgr}\text{-}S}(\mathcal{E}, \mathcal{E})$. By Lemma 7.14, $\mathrm{Ext}^0(\mathcal{E}, \mathcal{E}) = k$ and $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) = 0$. Corollary 6.2 now implies that $T_{[\mathcal{E}]}\mathcal{M}^s_S$ has the required dimension $1 - \chi(\mathcal{E}, \mathcal{E}) = r^2 + 3rc_1 + c_1^2 - 2r\chi + 1$. □

We want to examine when our moduli spaces are connected. Although we are mainly concerned with case of modules of rank one, the proof actually works for invariants $\{r, c_1, \chi\}$ for which $\mathcal{M}^{ss}(r, c_1, \chi) = \mathcal{M}^s(r, c_1, \chi)$ and the next lemma provides various cases where this is automatic.

Lemma 8.5. *Let $S \in \underline{\mathrm{AS}}_3$ and consider invariants $\{r, c_1, \chi\}$ with $-r < c_1 \leq 0$. Then $\mathcal{M}^{ss}(r, c_1, \chi) = \mathcal{M}^s(r, c_1, \chi)$, provided that* either (1) $c_1 = 0$ *and* $(r, \chi) = 1$ or (2) $c_1 \neq 0$ *and* $(r, c_1) = 1$.

Proof. In each case, use (6.1) to show that there cannot exist torsion-free modules $0 \neq \mathcal{F} \subsetneq \mathcal{M} \in \mathrm{qgr}\text{-}S$ for which $\mathrm{rk}(\mathcal{M})p_{\mathcal{F}} = \mathrm{rk}(\mathcal{F})p_{\mathcal{M}}$. □

Proposition 8.6. *Let $\mathcal{B}$ be an irreducible k-scheme of finite type, where* $\mathrm{char}(k) = 0$*, and fix a $\mathcal{B}$-flat family of algebras $S_{\mathcal{B}} \in \underline{\mathrm{AS}}_3$. Pick integers $\{r \geq 1, c_1, \chi\}$.*

(1) *Suppose that* $\mathcal{M}^s_{S_b}(r, c_1, \chi)$ *is nonempty for some* $b \in \mathcal{B}$*. Then* $\mathcal{M}^{ss}_{S_{b'}}(r, c_1, \chi)$ *is nonempty for every* $b' \in \mathcal{B}$*.*

(2) *Suppose that* $\mathcal{B}$ *is a smooth* k*-curve. Pick invariants* $\{r, c_1, \chi\}$ *for which* $\mathcal{M}^{ss}_{S_\mathcal{B}}(r, c_1, \chi) = \mathcal{M}^s_{S_\mathcal{B}}(r, c_1, \chi)$*. If* $\mathcal{M}^s_{S_b}(r, c_1, \chi)$ *is irreducible for some* $b \in \mathcal{B}$*, then* $\mathcal{M}^s_{S_{b'}}(r, c_1, \chi)$ *is irreducible for every* $b' \in \mathcal{B}$*.*

Proof. Set $\mathcal{M}^{ss} = \mathcal{M}^{ss}_{S_\mathcal{B}}(r, c_1, \chi)$ and $\mathcal{M}^{ss}_p = \mathcal{M}^{ss}_{S_p}(r, c_1, \chi) \cong \mathcal{M}^{ss} \otimes_\mathcal{B} k(p)$ for $p \in \mathcal{B}$ (and similarly for $\mathcal{M}^s$).

(1) The morphism $\mathcal{M}^{ss} \xrightarrow{\rho} \mathcal{B}$ is proper [Se, Theorem 4] and thus its image is closed in $\mathcal{B}$. On the other hand, Theorem 8.1 implies that $\rho|_{\mathcal{M}^s} : \mathcal{M}^s \to \mathcal{B}$ is flat; consequently its image is open in $\mathcal{B}$ [EGA, Théorème IV.2.4.6] and, by assumption, nonempty. Thus $\mathrm{Im}(\rho)$ is closed and contains a nonempty open subset of $\mathcal{B}$. Since $\mathcal{B}$ is irreducible, this implies that $\mathrm{Im}(\rho) = \mathcal{B}$.

(2) By Theorem 8.1 and the fact that $\mathcal{M}^{ss} = \mathcal{M}^s$, the morphism $\mathcal{M}^{ss} \xrightarrow{\rho} \mathcal{B}$ is flat. The morphism ρ is also proper; consequently, [Ha, Proposition III.8.7] implies that $p_*\mathcal{O}_{\mathcal{M}^{ss}}$ is a coherent subsheaf of a torsion-free $\mathcal{O}_\mathcal{B}$-module. Thus it is a vector bundle. We have $\mathrm{H}^0(\mathcal{O}_{\mathcal{M}^{ss}} \otimes k(b)) = \mathrm{H}^0(\mathcal{O}_{\mathcal{M}^{ss}_b}) = k(b)$ since the fibre of $\mathcal{M}^{ss}$ over b is irreducible and projective. Since $k \subseteq p_*\mathcal{O}_{\mathcal{M}^{ss}}$, it follows that the map $p_*\mathcal{O}_{\mathcal{M}^{ss}} \otimes k(b) \to \mathrm{H}^0(\mathcal{O}_{\mathcal{M}^{ss}} \otimes k(b))$ is surjective and so, by Theorem 4.3, the fibre of $p_*\mathcal{O}_{\mathcal{M}^{ss}}$ over b is 1-dimensional. Thus, $p_*\mathcal{O}_{\mathcal{M}^{ss}}$ is a line bundle. By [Ha, Corollary III.11.3] this implies that p has connected fibres. Since those fibres are also smooth they must be irreducible. □

By Corollary 7.17, $\mathcal{M}^{ss}(1, 0, \chi) \neq \emptyset$ for $\chi \leq 1$. In contrast, even over $\mathbf{P}^2$ it is a subtle question to determine when $\mathcal{M}^{ss}(r, c_1, \chi)$ is nonempty for $r > 1$ and that question has been studied in detail in [DL]. In the noncommutative case, the question is likely to be similarly subtle, although a number of positive results can be obtained by combining Proposition 8.6 with [DL, Théorème B]. Since that theorem is rather technical we merely note:

Corollary 8.7. *Assume that* $\mathcal{B}$ *is an irreducible* k*-scheme of finite type, where* $\mathrm{char}\, k = 0$*, and let* $S_\mathcal{B}$ *be a* $\mathcal{B}$*-flat family of algebras in* $\underline{\mathrm{AS}}_3$ *such that* $S_b \cong k[x, y, z]$*, for some* $b \in \mathcal{B}$*. If* $(r, c_1) = 1$*, then* $\mathcal{M}^s_{S_p}(r, c_1, \chi) \neq \emptyset$ *for all* $p \in \mathcal{B}$ *and all* $\chi \ll 0$*.*

Proof. By [DL, Théorème B], $\mathcal{M}^s_{S_b}(r, c_1, \chi) \neq \emptyset$ for $\chi \ll 0$. Now apply Proposition 8.6. □

Using [DL], one can give precise bounds in this corollary. As an illustration, by combining Corollary 8.7 with the computation in [DL, p.196] shows:

Example 8.8. In Corollary 8.7, $\mathcal{M}_{S_p}(20, 9, \chi) \neq \emptyset$ if $\chi \leq 24$.

8.2. **Rank One Modules.** Let $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$. In this subsection we examine $\mathcal{M}^s_S(1, c_1, \chi)$ in more detail and justify the assertions made at the end of Section 7. By Remark 7.9, there is no harm in assuming that $c_1 = 0$ and we always do so. As in the introduction we define $(\mathbf{P}_S \smallsetminus E)^{[n]}$ to be the subscheme of $\mathcal{M}^{ss}_S(1, 0, 1-n) = \mathcal{M}^s_S(1, 0, 1-n)$ parametrizing modules whose restrictions to E are line bundles. In other words, we are concerned with modules in the set $\mathcal{V}_S$ from Section 3. By Corollary 7.12, $(\mathbf{P}_S \smallsetminus E)^{[n]}$ is an open subscheme of $\mathcal{M}^{ss}_S(1, 0, 1-n)$. As we show next it is always nonempty and generically consists of line bundles.

There are two ways to prove the nonemptiness of $(\mathbf{P}_S \smallsetminus E)^{[n]}$. We will give a constructive proof. The other method is to prove inductively that the complement of $(\mathbf{P}_S \smallsetminus E)^{[n]}$ has dimension less than $2n = \dim \mathcal{M}_S^{ss}(1,0,1-n)$.

Proposition 8.9. *Let $S = S(E,\mathcal{L},\sigma) \in \underline{\mathrm{AS}}_3'$ and let $\epsilon = \epsilon(S)$ denote the minimal integer $m > 0$ (possibly $m = \infty$) for which $\mathcal{L}^{\sigma^m} \cong \mathcal{L}$ in* Pic *E.*

Then for every $0 \le n < \epsilon$, there exists a line bundle $\mathcal{M}_n$ in qgr-*S with first Chern class $c_1 = 0$ and Euler characteristic $\chi = 1 - n$.*

Proof. Clearly we may take $\mathcal{M}_0 = \mathcal{O}$, so suppose that $\epsilon > n > 0$. We will define $\mathcal{M}_n$ by constructing a short exact sequence

$$0 \to \mathcal{M}_n \to \mathcal{O}(1) \oplus \mathcal{O}(n) \to \mathcal{O}(n+1) \to 0. \tag{8.2}$$

Note that $\mathrm{H}^0(\mathcal{O}(m)) = \frac{1}{2}m^2 + \frac{3}{2}m + 1$ for $m \ge 0$ while $\mathrm{H}^i(\mathcal{O}(m)) = 0$, for $m, i \ge 1$, by Lemma 2.3(5). An elementary computation therefore shows that $\chi(\mathcal{M}_n) = 1-n$ and $c_1(\mathcal{M}_n) = 0$, as required.

By Lemma 2.5, $\mathcal{O}(m)|_E \cong \mathcal{L}_m^{\sigma^{-m}}$ for $m \ge 0$. Suppose first that $n > 1$. Since $\mathcal{L}$ is very ample, there exists an injection $\theta_1 : \mathcal{L}_{n-1}^{\sigma^{-(n-1)}} \hookrightarrow \mathcal{L}_n^{\sigma^{-(n)}}$. As E has dimension one, $\mathcal{K} = \mathrm{Coker}(\theta_1)$ is a sheaf of finite length. Since $\mathcal{K}$ is a homomorphic image of a very ample invertible sheaf, it is also a homomorphic image of $\mathcal{O}$. Let $\theta_2 : \mathcal{O} \twoheadrightarrow \mathcal{K}$ be the corresponding map. Since $\mathrm{H}^1(E, \mathcal{L}_{n-1}^{\sigma^{-(n-1)}}) = 0$, the map θ_2 lifts to a map $\mathcal{O}_E \to \mathcal{L}_n^{\sigma^{-(n)}}$. Combined with θ_1 we have therefore constructed a surjection $\theta' : \mathcal{O}|_E \oplus \mathcal{O}(n-1)|_E \twoheadrightarrow \mathcal{O}(n)|_E$. When $n = 1$ the existence of θ' is just the standard assertion that the very ample invertible sheaf $\mathcal{L}^{\sigma^{-1}}$ is a homomorphic image of $\mathcal{O}_E \oplus \mathcal{O}_E$. Shifting θ' by 1 gives (for all $n \ge 1$) the surjection

$$\theta : \mathcal{O}(1) \oplus \mathcal{O}(n) \twoheadrightarrow \mathcal{O}(1)|_E \oplus \mathcal{O}(n)|_E \twoheadrightarrow \mathcal{O}(n+1)|_E.$$

By Lemma 2.3(5), the map $\mathrm{H}^0(\mathcal{O}(m)) \to \mathrm{H}^0(\mathcal{O}(m)|_E)$ is surjective for all $m \in \mathbb{Z}$ and so θ lifts to a morphism $\phi : \mathcal{O}(1) \oplus \mathcal{O}(n) \to \mathcal{O}(n+1)$. Let $\mathcal{F} = \mathrm{Coker}(\phi)$ and suppose that $\mathcal{F} \neq 0$. By construction, $\mathcal{F}|_E = 0$ and so, by Lemma 6.15, $\mathcal{F}$ has finite length. Thus $\epsilon < \infty$, by [ATV2, Proposition 7.8]. Since $\mathcal{F}$ is a homomorphic image of $\mathcal{O}(n+1)/\phi(\mathcal{O}(n))$, the shift $\mathcal{F}(-n)$ is a homomorphic image of a line module, in the notation of [ATV2]. By [ATV2, Proposition 7.8] and in the notation of that result, $\eta = \epsilon(\mathcal{F}(-n)) \ge \epsilon$ and hence, by [ATV2, Proposition 6.7(i)], $\mathcal{F}(-n)$ has a minimal resolution of the form

$$\mathcal{O}(-\eta - 1) \to \mathcal{O}(-1) \oplus \mathcal{O}(-\eta) \to \mathcal{O} \to \mathcal{F}(-n) \to 0.$$

By Serre duality and [ATV2, Lemma 4.8], $\mathrm{H}^i(\mathcal{F}(m)) = 0$ for all $i > 0$ and $m \in \mathbb{Z}$. A simple computation using Hilbert series then ensures that $c_1(\mathcal{F}) = 0$ but $\chi(\mathcal{F}) = \mathrm{H}^0(\mathcal{F}) = \eta$. By the argument of the first paragraph of this proof, this implies that $\mathcal{M}_n = \mathrm{Ker}(\phi)$ satisfies $c_1(\mathcal{M}_n) = 0$ but $\chi(\mathcal{M}_n) = 1 - n + \eta > 1$. By Corollary 6.6(1), this contradicts Remark 6.5.

Thus $\mathcal{F} = 0$ and we have constructed the desired exact sequence (8.2). $\square$

When $\epsilon \le n$ we need to combine Proposition 8.9 with the idea behind Corollary 7.17 in order to show that $(\mathbf{P}_S \smallsetminus E)^{[n]} \neq \emptyset$.

Corollary 8.10. *Let $S = S(E,\mathcal{L},\sigma) \in \underline{\mathrm{AS}}_3'$. Then, for any $m \ge 0$, there exists a torsion-free, rank one module $\mathcal{M}_m \in$ qgr-S such that $c_1(\mathcal{M}_m) = 0$, $\chi(\mathcal{M}_m) = 1-m$ and $\mathcal{M}|_E$ is torsion-free. In particular, $(\mathbf{P}_S \smallsetminus E)^{[m]}$ is nonempty.*

Proof. Define ϵ as in Proposition 8.9. When $\epsilon = 1$, [ATV2, Theorem 7.3] implies that qgr-$S \simeq \mathrm{coh}(\mathbf{P}^2)$, where the result is standard. So we may assume that $\epsilon > 1$. If $0 \leq n < \epsilon$, then the module $\mathcal{M}_n$ defined by Proposition 8.9 has the required properties and so we may also assume that $\epsilon < \infty$. For $m \geq \epsilon$, write $m = r\epsilon + n$, with $0 \leq n < \epsilon$.

As $\mathcal{M}_n$ is a rank one torsion-free module, $\mathcal{M}_n \hookrightarrow r^{-1}\mathcal{O}_S$, for some nonzero $r \in S$; equivalently $\mathcal{M}_n \hookrightarrow \mathcal{O}(t)$, for some $t \geq 0$. By [ATV2, Lemma 7.17 and Equation 2.25], there exist infinitely many nonisomorphic simple modules $\mathcal{N}_\alpha \cong \mathcal{O}(t)/\mathcal{J}_\alpha \in$ qgr-S with $\epsilon(\mathcal{N}_\alpha) = \epsilon$. As in the proof of [ATV2, Theorem 7.3(ii)], $\bigcap \mathcal{J}_\alpha = 0$ and so $\mathcal{M}_n \cap \bigcap_\alpha (\mathcal{J}_\alpha) = 0$. We may therefore chose $\alpha_1, \dots \alpha_r$ such that, if $\mathcal{R}_m = \mathcal{M}_n \cap (\bigcap_{i=1}^r \mathcal{J}_{\alpha_i})$, then $\mathcal{M}_n/\mathcal{R}_m \cong \bigoplus_{i=1}^r \mathcal{N}_{\alpha_i}$. Set $\mathcal{M}_m = \mathcal{R}_m$. As in the proof of Proposition 8.9, $c_1(\mathcal{N}_{\alpha_i}) = 0$ and $\chi(\mathcal{N}_{\alpha_i}) = \epsilon$. Thus, $c_1(\mathcal{M}_m) = 0$ and $\chi(\mathcal{M}_m) = 1 - n - r\epsilon = 1 - m$ for each m. Finally, [ATV2, Proposition 7.7] implies that $\mathcal{N}_\alpha|_E = 0$ and so $(\mathcal{M}_m)|_E \cong (\mathcal{M}_n)|_E$ is torsion-free. $\square$

We do not know whether there exist line bundles $\mathcal{M}_m \in$ qgr-S with $c_1(\mathcal{M}_m) = 0$ and $\chi(\mathcal{M}_m) = 1 - m$, for $m \geq \epsilon$. Such modules clearly do not exist when qgr-$S \simeq \mathrm{coh}(\mathbf{P}^2)$ (where $\epsilon = 1$) and we suspect that they do not exist when $\epsilon > 1$.

Combining the last several results gives the following detailed description of the spaces $\mathcal{M}_S^{ss}(1, 0, 1-n)$ and $(\mathbf{P}_S \smallsetminus E)^{[n]}$, and proves Theorem 1.1 from the introduction.

Theorem 8.11. *Let $S = S(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3(k)$, where* $\mathrm{char}(k) = 0$. *Then*

1. *$\mathcal{M}_S^{ss}(1, 0, 1-n)$ is a smooth, projective, fine moduli space for equivalence classes of rank one torsion-free modules $\mathcal{M} \in$ qgr-S with $c_1(\mathcal{M}) = 0$ and $\chi(\mathcal{M}) = 1 - n$. Moreover,* $\dim \mathcal{M}_S^{ss}(1, 0, 1-n) = 2n$.
2. *$(\mathbf{P}_S \smallsetminus E)^{[n]}$ is a nonempty open subspace of $\mathcal{M}_S^{ss}(1, 0, 1-n)$.*
3. *Let $\epsilon = \epsilon(S)$ be defined as in Proposition 8.9. If $n < \epsilon$, then $(\mathbf{P}^2 \smallsetminus E)^{[n]}$ parametrizes line bundles with invariants $\{1, 0, 1-n\}$.*

Proof. Part (1) follows from Corollaries 7.17, 8.3 and 8.4. Part (2) then follows from Corollaries 7.12 and 8.10.

(3) Let $\mathcal{M} \in$ qgr-S be a torsion-free module, with torsion-free restriction to E and invariants $\{1, 0, 1-n\}$. If $\mathcal{M}$ is not a line bundle then M is not reflexive, by Lemma 3.11. Since $\mathcal{M}^{**}/M$ has finite length it must, by hypothesis, have a composition series of simple modules $\mathcal{N}_i$ with $\epsilon(\mathcal{N}_i) \geq \epsilon$. Just as in the penultimate paragraph of the proof of Proposition 8.9, this forces $\chi(\mathcal{M}^{**}) \geq 1 - n + \epsilon > 1$, which is impossible. Thus $\mathcal{M}$ is a line bundle. $\square$

When S is a deformation of the polynomial ring $k[x, y, z]$, we can say even more about $(\mathbf{P}_S \smallsetminus E)^{[n]}$. This next result completes the proof of Theorem 1.3 from the introduction.

Theorem 8.12. *Let $\mathcal{B}$ be a smooth curve defined over a field k of characteristic zero and let $S_\mathcal{B} = S_\mathcal{B}(E, \mathcal{L}, \sigma) \in \underline{\mathrm{AS}}'_3$ be a flat family of algebras such that $S_p = k[x, y, z]$ for some point $p \in \mathcal{B}$. Set $S = S_b$, for any point $b \in \mathcal{B}$. Then:*

1. *Both $\mathcal{M}_S^{ss}(1, 0, 1-n)$ and $(\mathbf{P}_S \smallsetminus E)^{[n]}$ are irreducible and hence connected.*
2. *$\mathcal{M}_S^{ss}(1, 0, 1-n)$ is a deformation of the Hilbert scheme $(\mathbf{P}^2)^{[n]}$, with its subspace $(\mathbf{P}_S \smallsetminus E)^{[n]}$ being a deformation of $(\mathbf{P}^2 \smallsetminus E)^{[n]}$.*

Proof. The fibre of $\mathcal{M}^{ss}_S(1,0,1-n)$ at the special point p is irreducible by [HL, Example 4.5.10]. Thus both $\mathcal{M}^{ss}_S(1,0,1-n)$ and its open subvariety $(\mathbf{P}_S \smallsetminus E)^{[n]}$ are irreducible by Proposition 8.6, proving (1). Part (2) follows from Theorem 8.1 the final assertion of Corollary 7.17. $\square$

There are no (commutative!) deformations of $(\mathbf{P}^2)^{[1]} = \mathbf{P}^2$ and so $\mathcal{M}^{ss}_S(1,0,0)$ must equal $\mathbf{P}^2$ in Theorem 8.12. In fact it is not difficult to prove directly that

$$\mathcal{M}^{ss}_S(1,0,0) \cong (\mathbf{P}^2)^{[1]} = \mathbf{P}^2 \qquad \text{for all} \quad S \in \underline{\mathrm{AS}}_3. \tag{8.3}$$

The proof is left to the interested reader.

8.3. **Framed Modules for the Homogenized Weyl algebra.** As we show in this subsection, the methods developed in this paper give a quick proof that the bijections of [BW1] and their generalizations in [KKO] do come from moduli space structures. This will prove Proposition 1.8 from the introduction.

In this subsection we fix $k = \mathbb{C}$ and let $U = \mathbb{C}\{x,y,z\}/(xy - yx - z^2, z\,\text{central})$ denote the homogenized Weyl algebra, as in (2.6). In this case, rather than restrict to the curve $E = \{z^3 = 0\}$ as we have done previously, we will use the projection $U \twoheadrightarrow U/Uz = \mathbb{C}[x,y]$ to identify $\mathrm{coh}(\mathbf{P}^1)$ with a subcategory of qgr-U. Fix $r \geq 1$ and $\chi \leq 1$. Following [KKO], if R is a commutative algebra, a *framed* torsion-free module in qgr-U_R is a rank r torsion-free object $\mathcal{E}$ of qgr-U_R equipped with an isomorphism $\mathcal{E}|_{\mathbf{P}^1} = \mathcal{O}^r_{\mathbf{P}^1} \otimes R$. (This is a different notion of framing from that considered for complexes on page 41.) A homomorphism $\theta : \mathcal{E} \to \mathcal{F}$ between two such objects is a homomorphism in qgr-U_R that induces a scalar multiplication on $\mathcal{E}|_{\mathbf{P}^1} = \mathcal{O}^r_{\mathbf{P}^1} = \mathcal{F}|_{\mathbf{P}^1}$ under the two framings. Let $\underline{\mathcal{M}}^{\mathrm{fr}}_U(r,0,\chi)$ denote the moduli functor of framed torsion-free modules in qgr-U of rank r and Euler characteristic χ.

Let $Z = \mathfrak{gl}_n \times \mathfrak{gl}_n \times M_{r,n} \times M_{n,r}$ be the vector space of quadruples of matrices of the prescribed sizes and set

$$V = \big\{(b_1, b_2, j, i) \in Z \;\big|\; [b_1, b_2] + ij + 2\mathbf{1}_{n,n} = 0\big\}$$

There is a natural action of $\mathrm{GL}(n)$ on V.

Proposition 8.13. *Let U be the homogenized Weyl algebra. Then the quotient $V/\!\!/\,\mathrm{GL}(n)$ represents the moduli functor $\underline{\mathcal{M}}^{\mathrm{fr}}_U(r,0,1-n)$.*

Proof. The proof amounts to using Theorem 5.8 to show that the constructions of [BW1, KKO] work in families.

We first use the construction of [KKO, Theorem 6.7] to define a map $V \to \underline{\mathcal{M}}^{\mathrm{fr}}_U(r,0,1-n)$. Given a commutative $\mathbb{C}$-algebra R and $(b_1,b_2,j,i) \in V(R)$, define a complex

$$\mathbf{K}:\ \mathcal{O}_{U_R}(-1)^n \xrightarrow{A} \mathcal{O}^n_{U_R} \oplus \mathcal{O}^n_{U_R} \oplus \mathcal{O}^r_{U_R} \xrightarrow{B} \mathcal{O}_{U_R}(1)^n \tag{8.4}$$

in qgr-U_R, where

$$A = \begin{pmatrix} \mathbf{1}_{n,n} \\ \mathbf{0}_{n,n} \\ \mathbf{0}_{r,n} \end{pmatrix} \cdot x + \begin{pmatrix} \mathbf{0}_{n,n} \\ \mathbf{1}_{n,n} \\ \mathbf{0}_{r,n} \end{pmatrix} \cdot y + \begin{pmatrix} b_1 \\ b_2 \\ j \end{pmatrix} \cdot z$$

and $B = x \cdot (\mathbf{0}_{n,n}\ \ \mathbf{1}_{n,n}\ \ \mathbf{0}_{n,r}) + y \cdot (-\mathbf{1}_{n,n}\ \ \mathbf{0}_{n,n}\ \ \mathbf{0}_{n,r}), +z \cdot (-b_2\ \ b_1\ \ i)$.

By [KKO, Theorem 6.7], $\mathbf{K} \otimes \mathbb{C}(p)$ is a torsion-free monad for every $p \in \operatorname{Spec} R$ and so $\mathbf{K}$ is a torsion-free monad in the sense of Definition 5.2. Thus Theorem 5.8 implies that $\mathrm{H}^0(\mathbf{K})$ is an R-flat family of torsion-free objects of qgr-U_R. It follows

that $\mathrm{H}^0(\mathbf{K})|_{\mathbf{P}^1} = \mathrm{H}^0(\mathbf{K}|_{\mathbf{P}^1})$ and so $\mathrm{H}^0(\mathbf{K})$ comes equipped with a framing ϕ. We thus obtain a $\mathrm{GL}(n)$-invariant map $V \to \underline{\mathcal{M}}^{\mathrm{fr}}_U(r,0,1-n)$.

We next show that the map of functors $\Psi : V/\mathrm{GL}(n) \to \underline{\mathcal{M}}^{\mathrm{fr}}_U(r,0,1-n)$ is an étale local isomorphism. Using Lemma 5.11, it follows from the proof of [KKO, Theorem 6.7] that Ψ is injective, so it suffices to prove étale local surjectivity.

Observe that if $(\mathcal{E},\phi)$ is a framed torsion-free object of qgr-U, then $\mathcal{E}$ is μ-semistable. To see this, suppose that $\mathcal{F} \subset \mathcal{E}$ de-semistabilizes $\mathcal{E}$. We may enlarge $\mathcal{F}$ if necessary and assume that $\mathcal{E}/\mathcal{F}$ is torsion-free. By Lemma 2.6 $\mathcal{F}|_{\mathbf{P}^1} \subset \mathcal{E}|_{\mathbf{P}^1} = \mathcal{O}^r_{\mathbf{P}^1}$ and so

$$c_1(\mathcal{F}) = c_1(\mathcal{F}|_{\mathbf{P}^1}) \leq 0 = c_1(\mathcal{E}|_{\mathbf{P}^1}) = c_1(\mathcal{E}).$$

Thus, $\mathcal{F}$ cannot de-semistabilize $\mathcal{E}$. The μ-semistability of $\mathcal{E}$ then implies that $\mathrm{Hom}(\mathcal{E},\mathcal{E}|_{\mathbf{P}^1}) = 0$, i.e. that framed torsion-free objects of qgr-U are rigid. By Theorem 4.3 the same result follows in families.

Lemma 6.4 implies that if $(\mathcal{E},\phi)$ is an R-flat family of framed torsion-free objects of qgr-U_R then $\mathcal{E}$ satisfies the Vanishing Condition 5.4. Theorem 5.8 then implies that $\mathcal{E} = \mathrm{H}^0(\mathbf{K})$ for some torsion-free monad $\mathbf{K}$. After an étale base change we may assume $\mathbf{K}$ is a monad of the form (8.4). In order to show that Ψ is surjective, it remains to show that this monad $\mathbf{K}$ is actually isomorphic to a monad coming from a quadruple $(b_1, b_2, j, i) \in V(R)$. This is immediate from the linear algebra of [KKO, Theorem 6.7].

As in [KKO, Theorem 6.7], the action of $\mathrm{GL}(n)$ on V is free, and so the étale slice theorem implies that the map $V \to V/\!\!/\mathrm{GL}(n)$ is a principle $\mathrm{GL}(n)$-bundle. But, since families of framed objects are rigid, the fibre product $V \times_{\underline{\mathcal{M}}^{\mathrm{fr}}} U$ is easily checked to be a principal $\mathrm{GL}(n)$-bundle for any $\mathbb{C}$-scheme U. Thus $V \to \underline{\mathcal{M}}^{\mathrm{fr}}_U(r,0,1-n)$ is also a principal $\mathrm{GL}(n)$-bundle and so $V/\!\!/\mathrm{GL}(n)$ represents $\underline{\mathcal{M}}^{\mathrm{fr}}_U(r,0,1-n)$. $\square$

9. Poisson Structure and Higgs Bundles

Throughout this section we assume that $k = \mathbb{C}$ and $S = \mathrm{Skl}(E,\mathcal{L},\sigma)$; thus E is an elliptic curve. Fix integer invariants $\{r \geq 1, c_1, \chi\}$ and write $\mathcal{M}^s = \mathcal{M}^s_S(r,c_1,\chi)$ for the corresponding smooth moduli space, as defined by Theorem 7.10.

The main aim of this section is to prove that $\mathcal{M}^s$ is Poisson and that the moduli space $(\mathbf{P}_S \smallsetminus E)^{[n]}$ is symplectic, thereby proving Theorems 1.4 and 1.9 from the introduction. We also take a first step towards relating our moduli spaces to integrable systems by describing $\mathcal{M}^s$ as the moduli for meromorphic Higgs bundles with structure group the centrally extended current group on E of Etingof-Frenkel [EF]. We achieve all these results by first observing that the moduli spaces of Kronecker complexes on S correspond to the moduli spaces for their restrictions to E, which we call σ-Kronecker complexes. This reduces the study of $\mathcal{M}^s$ to a purely commutative question and the desired Poisson structure then follows from results in [Pl].

We begin by discussing σ-Kronecker complexes.

Definition 9.1. Set $\overline{\mathcal{L}} = (\sigma^{-1})^*\mathcal{L}$. A *family of σ-Kronecker complexes parametrized by a $\mathbb{C}$-scheme U* is a complex on $U \times E$ of the form

$$\mathbf{K}: \ V_{-1} \otimes \mathcal{L}^* \xrightarrow{A} V_0 \otimes \mathcal{O}_E \xrightarrow{B} V_1 \otimes \overline{\mathcal{L}} \tag{9.1}$$

for some vector bundles V_{-1}, V_0 and V_1 on U. By Lemma 2.5, a family of σ Kronecker complexes (9.1) is the same as a complex of the form

$$V_{-1} \otimes \mathcal{O}_B(-1) \to V_0 \otimes \mathcal{O}_B \to V_1 \otimes \mathcal{O}_B(1), \qquad \text{where } B = S/gS. \tag{9.2}$$

We carry over the definitions of the invariants of a σ-Kronecker complex, σ-monad, etc. exactly as they appear for Kronecker complexes in qgr-S in Section 6.1. In particular, the definition of (semi)stability for a σ-Kronecker complex refers only to σ-Kronecker subcomplexes. A *σ-monad* is a σ-Kronecker complex (9.1) for which the map A is injective and the map B is surjective. A σ-Kronecker complex (or, as before, Kronecker complex) is *framed* if it comes equipped with an isomorphism $V_0 = \mathcal{O}^n$.

Proposition 9.2. *The moduli functors (stacks) for families of (normalized, semistable, stable, framed) Kronecker complexes in* qgr-S *and families of (normalized, semistable, stable, framed) σ-Kronecker complexes on E are isomorphic.*

Proof. Use (9.2) as the definition of a σ-Kronecker complex and consider the identities

$$\mathrm{Hom}_{\text{qgr-}S}(\mathcal{O}(n), \mathcal{O}(n+i)) = \mathrm{Hom}_E(\mathcal{O}(n)|_E, \mathcal{O}(n+i)|_E) \tag{9.3}$$

for $i = 0, 1, 2$. When $i = 1$, the identity gives the equivalence of diagrams of the form (6.18) over S and (9.1) over B. Then (9.3) for $i = 2$ gives the equivalence for the complexes among those diagrams, while (9.3) for $i = 0$ provides the equivalence of maps between these complexes. □

Corollary 9.3. *If* $\mathbf{K}$ *is a family of geometrically semistable σ-Kronecker complexes, then* $\mathbf{K}$ *is a σ-monad.*

Proof. Suppose that $\operatorname{Spec} R$ is the parameter scheme for $\mathbf{K}$. By Proposition 9.2, $\mathbf{K}$ lifts to a family $\widetilde{\mathbf{K}}$ of geometrically semistable Kronecker complexes in qgr-S_R. For every $p \in \operatorname{Spec} R$, we have $\mathbf{K} \otimes \mathbb{C}(p) = \widetilde{\mathbf{K}} \otimes \mathbb{C}(p)|_E$. By Corollary 6.19, $\widetilde{\mathbf{K}} \otimes \mathbb{C}(p)$ is therefore a torsion-free monad in qgr-$S_{\mathbb{C}(p)}$. Surjectivity of $B_{\mathbf{K}\otimes\mathbb{C}(p)}$ then follows from that of $B_{\widetilde{\mathbf{K}}\otimes\mathbb{C}(p)}$. Moreover, $\operatorname{coker}(A_{\widetilde{\mathbf{K}}\otimes\mathbb{C}(p)})$ is torsion-free and so Lemma 2.6 implies that $A_{\mathbf{K}\otimes\mathbb{C}(p)}$ is injective. □

9.1. **Construction of Poisson Structure.** The aim of this subsection is to prove the following result, which describes the Poisson structures of our moduli spaces.

Theorem 9.4. *Let $S = \mathrm{Skl}(E, \mathcal{L}, \sigma)$ be the Sklyanin algebra defined over $\mathbb{C}$ and pick integers $\{r \geq 1, c_1, \chi\}$. Then:*

1. *$\mathcal{M}^s_S(r, c_1, \chi)$ admits a Poisson structure.*
2. *Fix a vector bundle H on E and let $\mathcal{M}_H$ denote the smooth locus of the subvariety of $\mathcal{M}^s_S(r, c_1, \chi)$ that parametrizes those $\mathcal{E}$ for which $\mathcal{E}|_E \cong H$. Then $\mathcal{M}_H$ is an open subset of a symplectic leaf of the Poisson space.*

Remark 9.5. This result proves Theorem 1.9 from the introduction. Combined with Corollary 6.6(3) it also proves Theorem 1.4.

We begin by explaining the strategy for the proof. By Remark 7.9 we can and will assume that $-r < c_1 \leq 0$ (this may replace H by some shift $H(m)$). By Proposition 9.2 it then suffices to prove the existence of a Poisson structure on the corresponding moduli space Σ of geometrically stable σ-monads. As we will

explain, this essentially follows from work of Polishchuk [Pl] (see Proposition 9.9), although we will first need to set up the appropriate framework to apply his results.

Fix a positive integer n and let $\mathcal{M}''$ denote the moduli stack parametrizing data $(\mathcal{E}_i, \phi_i)$ consisting of n-tuples of vector bundles $\mathcal{E}_i$ on E and maps $\phi_i : \mathcal{E}_{i+1} \to \mathcal{E}_i$. There is a closed substack $\mathcal{M}'$ of $\mathcal{M}''$ parametrizing those $(\mathcal{E}_{i+1}, \phi_i)$ for which $\phi_i \circ \phi_{i+1} = 0$ for all i; in other words, $\mathcal{M}'$ parametrizes *complexes* of vector bundles. If $(\mathcal{F}_i)$ is an n-tuple of E-vector bundles, let $\mathcal{M} = \mathcal{M}(\mathcal{F}_1, \dots, \mathcal{F}_n)$ denote the locally closed substack of $\mathcal{M}'$ parametrizing data $(\mathcal{E}_i, \phi_i)$ for which $\mathcal{E}_i \cong \mathcal{F}_i$ for all i. As we will show, it follows from [Pl] that there is Poisson structure on an open substack of $\mathcal{M}'$ and this restricts to a Poisson structure on the smooth locus of $\mathcal{M}$. Since Σ is an open substack of $\mathcal{M}(\overline{\mathcal{L}}^{\,d_1}, \mathcal{O}_E^n, (\mathcal{L}^*)^{d_{-1}})$, in the notation of (7.2), this will give the desired Poisson structure on $\mathcal{M}^s(r, c_1, \chi)$.

Our first task is to construct a map Ψ'' on (co)tangent spaces of $\mathcal{M}''$, for which we use following concrete description of those spaces. Let $(\mathcal{E}_i, \phi_i)$ be data defining a point of $\mathcal{M}''$. The discussion of [Pl, Section 1], shows that the tangent space to $\mathcal{M}''$ at $(\mathcal{E}_i, \phi_i)$ is the first hypercohomology $\mathbf{H}^1(C)$ of a complex $C = C(\mathcal{E}_i, \phi_i)$ defined as follows: C is concentrated in degrees 0 and 1 with $C^0 = \bigoplus \underline{\mathrm{End}}(\mathcal{E}_i)$ and $C^1 = \bigoplus \underline{\mathrm{Hom}}(\mathcal{E}_{i+1}, \mathcal{E}_i)$. The differential d sends $(e_i) \in C^0$ to $(e_i\phi_i - \phi_i e_{i+1}) \in C^1$.

We will actually be interested in the open substack $\mathcal{M}''_0$ of the smooth locus of $\mathcal{M}''$ that consists of points $(\mathcal{E}_i, \phi_i)$ for which the hypercohomology of $C = C(\mathcal{E}_i, \phi_i)$ satisfies

$$\mathbf{H}^2(C) = 0 \text{ and } \mathbf{H}^0(C) = \mathbb{C}. \tag{9.4}$$

This substack parametrizes objects that have only scalar endomorphisms. The tangent sheaf of $\mathcal{M}''_0$ is a locally free sheaf, the fibre of which over $(\mathcal{E}_i, \phi_i)$ is $\mathbf{H}^1\big(C(\mathcal{E}_i, \phi_i)\big)$. Let $\mathcal{M}'_0$ denote the smooth locus of $\mathcal{M}' \cap \mathcal{M}''_0$ and similarly let $\mathcal{M}_0$ denote the smooth locus of $\mathcal{M} \cap \mathcal{M}''_0$.

We next construct a map $\Psi'' : T^*\mathcal{M}''|_{\mathcal{M}'_0} \to T\mathcal{M}''|_{\mathcal{M}'_0}$. Let $C = C(\mathcal{E}_i, \phi_i)$. The dual complex $C^\vee[-1]$ of C is the complex $\bigoplus \underline{\mathrm{Hom}}(\mathcal{E}_i, \mathcal{E}_{i+1}) \xrightarrow{d^*} \bigoplus \underline{\mathrm{End}}(\mathcal{E}_i)$, again concentrated in degrees 0 and 1, with differential $d^* : (f_i) \mapsto (\phi_i f_i - f_{i-1}\phi_{i-1})$. By Serre duality, $T^*_{(\mathcal{E}_i,\phi_i)}\mathcal{M}'' \cong \mathbf{H}^1(C^\vee[-1])$. We define maps $\psi_k : (C^\vee[-1])^k \to C^k$ by

$$\psi_0(f_i) = \big((-1)^{i+1}(\phi_i f_i - f_{i-1}\phi_{i-1})\big) \qquad \text{and} \qquad \psi_1 = 0.$$

We define a second pair of maps $\overline{\psi}_k : C^\vee[-1]^k \to C^k$ by

$$\overline{\psi}_0 = 0 \qquad \text{and} \qquad \overline{\psi}_1 : (e_i) \mapsto \big((-1)^{i+1}(e_i\phi_i + \phi_i e_{i+1})\big).$$

The reason for working with $\mathcal{M}'_0$ rather than $\mathcal{M}''_0$, is that these maps define maps of complexes. Indeed, an elementary calculation, using the fact that $\phi_i\phi_{i+1} = 0$, gives the following result. (See also [Pl, Theorem 2.1].)

Lemma 9.6. *If $(\mathcal{E}_i, \phi_i)$ lies in $\mathcal{M}'_0$, then:*

1. *$\psi = (\psi_0, \psi_1)$ defines a morphism of complexes $\psi : C^\vee[-1] \to C$.*
2. *$\overline{\psi} = (\overline{\psi}_0, \overline{\psi}_1)$ defines a morphism of complexes $\overline{\psi} : C^\vee[-1] \to C$.*
3. *The map $h : \bigoplus \underline{\mathrm{End}}(\mathcal{E}_i) \to \bigoplus \underline{\mathrm{End}}(\mathcal{E}_i)$ defined by $h(e_i) = \big((-1)^i e_i\big)$ defines a homotopy between ψ and $\overline{\psi}$. Hence $\mathbf{H}^1(\psi) = \mathbf{H}^1(\overline{\psi})$.* □

As in [Pl, Section 2], the maps ψ and $\overline{\psi}$ globalize to give a map $\Psi'' : T^*\mathcal{M}''|_{\mathcal{M}'_0} \to T\mathcal{M}''|_{\mathcal{M}'_0}$ that, in the fibres over $(\mathcal{E}_i, \phi_i)$, is exactly the map $\mathbf{H}^1(\psi)$. The proof of [Pl, Theorem 2.1] also shows that Ψ'' is skew-symmetric.

We next want to show that Ψ'' factors through maps

$$\Psi' : T^*\mathcal{M}_0' \to T\mathcal{M}_0' \qquad \text{and} \qquad \Psi : T^*\mathcal{M}_0 \to T\mathcal{M}_0.$$

Let $(\mathcal{E}_i, \phi_i)$ define a point of $\mathcal{M}_0'$ and $C = C(\mathcal{E}_i, \phi_i)$. Define a complex B by setting $B^0 = \bigoplus \underline{\mathrm{End}}(\mathcal{E}_i) = C^0$ and $B^1 = \bigoplus \underline{\mathrm{Hom}}(\mathcal{E}_{i+2}, \mathcal{E}_i)$, with the zero differential. A simple calculation shows that there is a map of complexes $\Xi : C \to B$, where Ξ^0 is the identity and Ξ^1 maps $(\psi_i) \in C^1$ to $(\phi_i\psi_{i+1} + \psi_i\phi_{i+1})$.

Lemma 9.7. (1) $T_{(\mathcal{E}_i,\phi_i)}\mathcal{M}_0' = \ker\left[\mathbf{H}^1(C) \to \mathbf{H}^1(B^1)\right]$.

(2) *If $(\mathcal{E}_i, \phi_i)$ determines a point of $\mathcal{M}_0$, then $T_{(\mathcal{E}_i,\phi_i)}\mathcal{M}_0 = \ker(\mathbf{H}^1(\Xi))$.*

(3) *The cotangent spaces $T^*_{(\mathcal{E}_i,\phi_i)}\mathcal{M}_0'$ and $T^*_{(\mathcal{E}_i,\phi_i)}\mathcal{M}_0$ are the cokernels of the respective dual maps.*

Proof. Since the differential in B is zero,

$$\mathbf{H}^1(B) = \mathbf{H}^1(B^0) \oplus \mathbf{H}^1(B^1) = \left[\bigoplus \mathrm{H}^1(\underline{\mathrm{End}}(\mathcal{E}_i))\right] \oplus \left[\bigoplus \mathrm{Hom}(\mathcal{E}_{i+2}, \mathcal{E}_i)\right]. \tag{9.5}$$

A cocycle calculation shows that a class c corresponding to a first-order deformation $(\widetilde{\mathcal{E}}_i, \widetilde{\phi}_i) \in T_{(\mathcal{E}_i,\phi_i)}\mathcal{M}_0'$ lies in $\mathrm{Ker}\left[\mathbf{H}^1(C) \to \mathbf{H}^1(B^1)\right]$ if and only if $\widetilde{\phi}_i \circ \widetilde{\phi}_{i+1} = 0$ for all i. This proves part (1). Similarly, a calculation shows that $c \in \ker(\mathbf{H}^1(\Xi))$ if and only if both $\widetilde{\phi}_i \circ \widetilde{\phi}_{i+1} = 0$ for all i and every $\widetilde{\mathcal{E}}_i$ is a trivial deformation of $\mathcal{E}_i$, proving (2). Part (3) then follows by Serre duality. $\square$

Corollary 9.8. *$\Psi''|_{\mathcal{M}_0}$ factors through a map $\Psi : T^*\mathcal{M}_0 \to T\mathcal{M}_0$. Similarly, $\Psi''|_{\mathcal{M}_0'}$ factors through a map $\Psi' : T^*\mathcal{M}_0' \to T\mathcal{M}_0'$.*

Proof. We will just prove the first assertion, since the same proof works in both cases.

By Lemma 9.7, to prove that $\Psi''|_{\mathcal{M}_0'}$ factors through $T\mathcal{M}_0$ it suffices to show that the composite map

$$\mathbf{H}^1(C^\vee[-1]) \xrightarrow{\Psi''} \mathbf{H}^1(C) \xrightarrow{\mathbf{H}^1(\Xi)} \mathbf{H}^1(B) \tag{9.6}$$

is zero. To prove this we combine Lemma 9.6(3) with two simple observations: first, since ψ_1 is defined to be zero, the composite of (9.6) followed by projection onto $\bigoplus \mathrm{Hom}(\mathcal{E}_{i+2}, \mathcal{E}_2) = \mathbf{H}^1(B^1)$ is zero. Secondly, the homotopic map of complexes $\overline{\psi}$ is zero in cohomological degree 0 and so the composite of (9.6) followed by projection onto $\mathrm{H}^1(\underline{\mathrm{End}}(\mathcal{E}_i))$ is zero. It therefore follows from (9.5) that (9.6) is zero.

To obtain the factorization through $T^*\mathcal{M}_0'$ and $T^*\mathcal{M}_0$ we use the description from Lemma 9.7 in terms of the dual map $\mathbf{H}^1\big((B^1)^\vee[-1]\big) \to \mathbf{H}^1(C^\vee[-1])$. Taking the dual of (9.6) and using that $(\Psi'')^\vee = -\Psi''$, it follows that the factorization through cotangent spaces is equivalent to the vanishing of the dual to (9.6). $\square$

We now have the following consequence of Polishchuk's work.

Proposition 9.9. *The map Ψ' is a Poisson structure on $\mathcal{M}_0'$ while Ψ is a Poisson structure on $\mathcal{M}_0$.*

Proof. We only prove the first assertion since the second one follows from the same argument. This is really a special case of [Pl, Theorem 6.1], but we need to check that this particular result does fit into Polishchuk's framework.

We start by reinterpreting the map $\psi : C^\vee[-1] \to C$ from Lemma 9.6. Fix a component of $\mathcal{M}''$ and let $n_i = \mathrm{rk}(\mathcal{E}_i)$ be the corresponding ranks of the vector

bundles. Let $G = \prod \mathrm{GL}_{n_i}(\mathbb{C})$ and $V = \bigoplus \mathrm{Hom}(\mathbb{C}^{n_{i+1}}, \mathbb{C}^{n_i})$ with the natural G-action. Then $\mathcal{M}''$ can and will be identified with an open substack of the moduli stack of pairs (P, s) consisting of a principal G-bundle P and a section s of the associated bundle $P \times_G V$. Under this identification, the pair (P, s) maps to the data $(\mathcal{E}_i, \phi_i)$ where the sequence of vector bundles $\mathcal{E}_i$ is given by $P \times_G \bigoplus \mathbb{C}^{n_i}$ and the maps ϕ_i are determined by s.

Next, set $\mathfrak{g}_{n_i} = \mathrm{Lie}(\mathrm{GL}_{n_i}(\mathbb{C}))$ and $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathfrak{t}_{n_i} \in \mathrm{Sym}^2(\mathfrak{g}_{n_i})^{\mathfrak{g}_{n_i}}$ denote the dual of the trace pairing and set $\mathfrak{t} = \bigoplus (-1)^{i+1} \mathfrak{t}_{n_i} \in \mathrm{Sym}^2(\mathfrak{g})^{\mathfrak{g}}$. By abuse of notation we also use $\mathfrak{t}$ to denote the induced map $\mathfrak{g}^* \to \mathfrak{g}$. Associated to each $v \in V$ one has the map $d_v : \mathfrak{g} \to V$ defined by $d_v(X) = X \cdot v$ and this induces the map $\mathfrak{t} \circ d_v^* : V^* \to \mathfrak{g}$. We therefore obtain a map $\psi_0 = P \times_G (\mathfrak{t} \circ d_v^*) : P \times_G V^* \to P \times_G \mathfrak{g}$ associated to a pair (P, s). As in [Pl], this is precisely the map $\psi_0 : (C^\vee[-1])^0 \to C^0$ from Lemma 9.6. As before, define $\psi_1 : (C^\vee[-1])^1 \to C^1$ to be the zero map.

Sublemma 9.10. *If $v = (\mathcal{E}_i, \phi_i) \in \mathcal{M}'$, then $d_v \circ \mathfrak{t} \circ d_v^* = 0$.*

Proof of the sublemma. Although the proof is an elementary computation, we give it since the result is the basic hypothesis for [Pl, Theorem 6.1] (see [Pl, p.699]). Identify $V^* = \bigoplus \mathrm{Hom}(\mathbb{C}^{n_i}, \mathbb{C}^{n_{i+1}})$ by $\langle (\alpha_i), (\beta_j) \rangle = \sum_i \mathrm{Tr}(\alpha_i \beta_i)$ for $(\alpha_i) \in V$ and $(\beta_j) \in V^*$. Fix $(\alpha_i) \in V$ corresponding to a point in $\mathcal{M}'$. Then $d_{(\alpha_i)}(X_j) = (X_i \alpha_i - \alpha_i X_{i+1})$ for $(X_j) \in \mathfrak{g}$. Thus, $d^*_{(\alpha_i)}$ is defined by

$$\begin{aligned} d^*_{(\alpha_i)}(\beta_j)(X_k) &= (\beta_j)(d_{(\alpha_i)})(X_k) = \langle d_{(\alpha_i)}(X_k)\,,\, (\beta_j) \rangle \\ &= \textstyle\sum_i \mathrm{Tr}(X_i \alpha_i \beta_i) - \sum_i \mathrm{Tr}(\alpha_{i+1} X_{i+1} \beta_i) = \sum_i \mathrm{Tr}(X_i(\alpha_i \beta_i - \beta_{i-1}\alpha_{i-1})). \end{aligned}$$

This implies that $\mathfrak{t} \circ d^*_{(\alpha_i)}(\beta_j) = (-1)^{i+1}(\alpha_i \beta_i + \beta_{i-1}\alpha_{i-1})$. Finally,

$$d_{(\alpha_i)} \circ \mathfrak{t} \circ d^*_{(\alpha_i)}(\beta_j) = (-1)^{i+1}(\alpha_i\beta_i\alpha_i + \beta_{i-1}\alpha_{i-1}\alpha_i - \alpha_i\alpha_{i+1}\beta_{i+1} - \alpha_i\beta_i\alpha_i).$$

But $\alpha_i \alpha_{i+1} = 0$ by the definition of $\mathcal{M}'$. Thus $d_{(\alpha_i)} \circ \mathfrak{t} \circ d^*_{(\alpha_i)} = 0$. □

We return to the proof of the proposition. As in [Pl, p.699], skew-symmetry of the Poisson bracket follows from the sublemma, so it remains to prove the Jacobi identity. This is equivalent to the equation

$$\Psi'(\omega_1) \cdot \langle \Psi'(\omega_2), \omega_3 \rangle - \langle [\Psi'(\omega_1), \Psi'(\omega_2)], \omega_3 \rangle + \mathrm{cp}(1,2,3) = 0, \tag{9.7}$$

where the ω_i are local 1-forms on $\mathcal{M}'_0$ and $\mathrm{cp}(1,2,3)$ denotes cyclic permutations of the indices $\{1,2,3\}$. By Corollary 9.8, we may replace Ψ' by $\Psi''|_{\mathcal{M}'_0}$ in (9.7). The fact that (9.7) holds is now [Pl, Theorem 6.1], except that we have to work with $\mathcal{M}'$ rather than $\mathcal{M}''$ in order to ensure that the hypothesis of Sublemma 9.10 is valid. However, the proof of [Pl, Theorem 6.1] can be used without change to prove (9.7) and hence the proposition. □

It is now easy to complete the proof of Theorem 9.4.

Proof of Theorem 9.4. By Remark 7.9 we may assume that $-r < c_1 \leq 0$. By Propositions 6.20 and 9.2, it then suffices to prove the result for σ-monads rather than modules. By Lemma 2.5 and (7.2), such a σ-monad has the form

$$\mathcal{E} : \ \mathcal{E}_3 \xrightarrow{\phi_2} \mathcal{E}_2 \xrightarrow{\phi_1} \mathcal{E}_1, \quad \text{where } \mathcal{E}_3 \cong (\mathcal{L}^*)^{d_1},\ \mathcal{E}_2 \cong \mathcal{O}^n \text{ and } \mathcal{E}_1 \cong \overline{\mathcal{L}}^{\,d_1}. \tag{9.8}$$

Thus $\mathcal{M} = \mathcal{M}\big(\overline{\mathcal{L}}^{\,d_1}, \mathcal{O}^n_E, (\mathcal{L}^*)^{d-1}\big)$ is the moduli stack of σ-Kronecker complexes with the specified invariants and it has an open substack $\mathcal{M}^s$ that is the moduli

stack parametrizing the corresponding geometrically stable σ-monads. By Proposition 9.9, in order to prove part (1) of the theorem, it suffices to prove:

(i) $\mathcal{M}^s$ lies in the open substack of the smooth locus of $\mathcal{M}''$ consisting of points satisfying (9.4),
(ii) $\mathcal{M}^s$ lies in the smooth locus of $\mathcal{M}$ and
(iii) the Poisson structure on $\mathcal{M}$ induces one on $\mathcal{M}^s_S(r, c_1, \chi)$.

We first prove (i). Let $(\mathcal{E}_i, \phi_i)$ be the datum of a geometrically stable σ-monad $\mathcal{E}$ on E and set $C = C(\mathcal{E}_i, \phi_i)$. If Bun denotes the moduli stack of triples of vector bundles on E then Bun is smooth and there is a forgetful morphism $\mathcal{M}'' \to \text{Bun}$ sending $(\mathcal{F}_i, \phi_i)$ to $(\mathcal{F}_i)$. For the given datum $(\mathcal{E}_i, \phi_i)$, we have $\mathrm{Ext}^1(\mathcal{E}_{i+1}, \mathcal{E}_i) = 0$ for $i = 1, 2$. Thus Theorem 4.3(3) implies that $\mathcal{M}''$ is a vector bundle over Bun in a neighborhood of $(\mathcal{E}_i)$ and so $\mathcal{M}''$ is smooth at $(\mathcal{E}_i, \phi_i)$. By Serre duality and (9.8),

$$\mathbf{H}^2(C)^* = \mathbf{H}^0(C^\vee[-1]) = \ker\left[\bigoplus \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_{i+1}) \to \bigoplus \mathrm{End}(\mathcal{E}_i)\right] = 0.$$

On the other hand, $\mathbf{H}^0(C) = \ker\left[\bigoplus \mathrm{End}(\mathcal{E}_i) \xrightarrow{d} \bigoplus \mathrm{Hom}(\mathcal{E}_{i+1}, \mathcal{E}_i)\right]$ is the endomorphism ring $\mathrm{End}(\mathcal{E})$ of the σ-monad $\mathcal{E}$. Since $\mathcal{E}$ is stable, Lemma 7.14 and Proposition 9.2 imply that $\mathrm{End}(\mathcal{E}) = \mathbb{C}$. Thus (i) holds.

By Remark 7.11 and Proposition 9.2, $\mathcal{M}^s$ is isomorphic to the stack-theoretic quotient $[\mathcal{N}^s/\,\mathrm{GL}(H)]$ and so it is smooth at $(\mathcal{E}_i, \phi_i)$, by Remark 8.2. Thus (ii) holds.

Moreover, $\mathcal{M}^s$ comes equipped with a map $\alpha : \mathcal{M}^s \to \mathcal{M}^s(r, c_1, \chi)$. By the discussion at the beginning of Section 8, the étale slice theorem implies that α is étale locally of the form $U \times B\mathbb{C}^* \to U$ and so it induces isomorphisms of tangent and cotangent bundles under pullbacks. It follows that Ψ also defines a Poisson structure on $\mathcal{M}^s(r, c_1, \chi)$, proving (iii). Thus part (1) of the theorem is true.

(2) The strategy of the proof is the following. Let $\mathcal{M}_H$ denote the smooth locus of the locally closed subscheme of $\mathcal{M}^s_S(r, c_1, \chi)$ that parametrizes those $\mathcal{E}$ in qgr-S for which $\mathcal{E}|_E \cong H$. For such a module $\mathcal{E}$, it is awkward to directly relate $T_{\mathcal{E}}\mathcal{M}_H$ to the map Ψ. Instead, we construct a complex $D = D(\mathcal{E})$ together with a map $D \to C$ and use this to identify $\Psi'_{\mathcal{E}}$ with a map $\mathbf{H}^1(D^\vee[-1]) \to \mathbf{H}^1(D)$. It will then be easy to prove that $T_{\mathcal{E}}\mathcal{M}_H$ is exactly the image of this map and it will follow that Ψ' induces a nondegenerate map $T^*_{\mathcal{E}}\mathcal{M}_H \to T_{\mathcal{E}}\mathcal{M}_H$.

Thus, fix a vector bundle H on E and suppose that $\mathcal{E}$ is a σ-monad corresponding to a point of $\mathcal{M}_H$, as in (9.8). Then ϕ_2 makes $\mathcal{E}_3$ a subbundle of $\mathcal{E}_2$ and ϕ_1 is surjective. We write $\underline{\mathrm{End}}(\mathcal{E})$ for the sheaf of endomorphisms of the σ-monad $\mathcal{E}$.

Consider the complex

$$D: \ \bigoplus \underline{\mathrm{End}}(\mathcal{E}_i) \xrightarrow{d} \bigoplus \underline{\mathrm{Hom}}(\mathcal{E}_{i+1}, \mathcal{E}_i) \xrightarrow{\Xi^1} \bigoplus \underline{\mathrm{Hom}}(\mathcal{E}_{i+2}, \mathcal{E}_i),$$

where $\bigoplus \underline{\mathrm{End}}(\mathcal{E}_i)$ lies in degree zero, d is the differential in the complex C and Ξ^1 is the map defined before Lemma 9.7. There is a natural map of complexes $D \to C$ given by projection. As in [Pl, Lemma 3.1], a routine computation shows:

Fact 9.11. The cohomology of D satisfies $\mathrm{H}^i(D) = 0$ for $i \neq 0$ and $\mathrm{H}^0(D) = \underline{\mathrm{End}}(\mathcal{E})$.

A standard Čech calculation then gives $T_{(\mathcal{E}_i, \phi_i)}\mathcal{M}'_0 = \mathrm{H}^1(\underline{\mathrm{End}}(\mathcal{E}))$. It follows that there is a canonical isomorphism $T_{(\mathcal{E}_i, \phi_i)}\mathcal{M}'_0 \xrightarrow{\cong} \mathbf{H}^1(D)$.

The dual complex $D^\vee[-1]$ to D has the form

$$D^\vee[-1]: \quad \bigoplus \underline{\mathrm{Hom}}(\mathcal{E}_i, \mathcal{E}_{i+2}) \to \bigoplus \underline{\mathrm{Hom}}(\mathcal{E}_i, \mathcal{E}_{i+1}) \to \bigoplus \underline{\mathrm{End}}(\mathcal{E}_i).$$

We define a map $\psi_D : D^\vee[-1] \to D$ by taking $(\psi_D)_i = \psi_i$ for $i = 0, 1$ (where ψ is the map $C^\vee[-1] \to C$ defined above) and $(\psi_D)_{-1} = 0$. A straightforward computation shows that ψ_D is a homomorphism of complexes. It is clear from the construction that $\psi : C^\vee[-1] \to C$ factors through ψ_D and it follows that

$$\mathbf{H}^1(\psi_D) = (\Psi' : T^*_{\mathcal{E}}\mathcal{M}'_0 \longrightarrow T_{\mathcal{E}}\mathcal{M}'_0).$$

Thus, in order to show that Ψ' factors through $T_{\mathcal{E}}\mathcal{M}_H \subset T_{\mathcal{E}}\mathcal{M}'_0$ it suffices to show that $T_{\mathcal{E}}\mathcal{M}_H = \ker\left[\mathbf{H}^1(D) \to \mathbf{H}^1(\mathrm{Cone}(\psi_D))\right]$, where $\mathrm{Cone}(\psi_D)$ is the complex

$$\bigoplus \underline{\mathrm{Hom}}(\mathcal{E}_i, \mathcal{E}_{i+2}) \to \bigoplus \underline{\mathrm{Hom}}(\mathcal{E}_i, \mathcal{E}_{i+1}) \xrightarrow{(-d^*, \psi_0)} \left[\bigoplus \underline{\mathrm{End}}(\mathcal{E}_i)\right] \oplus \underline{\mathrm{End}}(\mathcal{E}),$$

with the right hand term in degree 0. By construction, $\mathrm{Cone}(\psi_D)$ is exact except at the right hand term. Define a map $\mathrm{Cone}(\psi_D) \longrightarrow [\bigoplus \underline{\mathrm{End}}(\mathcal{E}_i)] \oplus \underline{\mathrm{End}}(H)$ by

$$(A_3, A_2, A_1, F) \in \left[\bigoplus \underline{\mathrm{End}}(\mathcal{E}_i)\right] \oplus \underline{\mathrm{End}}(\mathcal{E}) \mapsto (A_1 + F_1, -A_2 + F, A_3 + F_3, \mathcal{H}^0(F)),$$

where F_i denotes the ith graded component of $F \in \underline{\mathrm{End}}(\mathcal{E})$. A routine check shows that this map is a quasi-isomorphism.

It follows that the sequence

$$\mathbf{H}^1(D^\vee[-1]) \xrightarrow{\Psi'} \mathbf{H}^1(D) \xrightarrow{\alpha} \left[\bigoplus \mathrm{H}^1(\underline{\mathrm{End}}(\mathcal{E}_i))\right] \oplus \mathrm{H}^1(\underline{\mathrm{End}}(H)) \tag{9.9}$$

is exact. Here, the map α takes the class of a first-order deformation of $\mathcal{E}$ as a complex to the classes of the associated first-order deformations of the $\mathcal{E}_i$ and of the middle cohomology of $\mathcal{E}$. Therefore, $\mathrm{Im}(\Psi')$ consists exactly of those first-order deformations of $(\mathcal{E}_i, \phi_i)$ such that the induced deformations of the $\mathcal{E}_i$ and H are trivial. This is exactly $T_{(\mathcal{E}_i,\phi_i)}\mathcal{M}_H$ and so Φ does factor through $T_{\mathcal{E}}\mathcal{M}_H$.

The skew-symmetry of Ψ' shows that the Serre dual to (9.9) is an exact sequence

$$\mathbf{H}^1((\mathrm{Cone}(\psi)^\vee[-1]) \to \mathbf{H}^1(D^\vee[-1]) \xrightarrow{\Psi'} \mathbf{H}^1(D).$$

It follows that Ψ' factors through

$$T^*_{(\mathcal{E}_i,\phi_i)}\mathcal{M}_H = \mathrm{coker}\left[\mathbf{H}^1((\mathrm{Cone}(\psi)^\vee[-1]) \to \mathbf{H}^1(D^\vee[-1])\right].$$

In particular, Ψ' induces a surjective map $T^*_{(\mathcal{E}_i,\phi_i)}\mathcal{M}_H \to T_{(\mathcal{E}_i,\phi_i)}\mathcal{M}_H$, which must therefore be an isomorphism. This completes the proof. □

9.2. **From Equivariant Bundles to Equivariant Higgs Bundles.** The treatment in this section will be informal, since it takes us rather far afield from the main results of the paper. However, inspired in part by constructions of integrable particle systems using current algebras (see [GN, Ne]), we want to indicate the relationship between the earlier results of this paper and a theory of Higgs bundles taking values in a centrally extended current group on the elliptic curve [EF].

Let $S = \mathrm{Skl}(E, \mathcal{L}, \sigma)$, for $k = \mathbb{C}$ and (without loss of generality) fix invariants $r, \dots, d_1$ by Notation 7.1. Fix also a vector bundle H on E of rank r and with determinant $\mathcal{L}^{\otimes d_{-1}} \otimes \overline{\mathcal{L}}^{\otimes -d_1}$. Fix the C^∞ complex vector bundle $\mathcal{V}$ on $E \times \mathbf{P}^1$, defined as the underlying C^∞ bundle of

$$\left((\mathcal{L}^*)^{d_{-1}} \boxtimes \mathcal{O}_{\mathbf{P}^1}\right) \oplus \left(H \boxtimes \mathcal{O}_{\mathbf{P}^1}(-1)\right) \oplus \left(\overline{\mathcal{L}}^{\,d_1} \boxtimes \mathcal{O}_{\mathbf{P}^1}(-2)\right) \ \in \ \mathrm{coh}(E \times \mathbf{P}^1).$$

Let $\mathbb{C}^*$ act on $E \times \mathbf{P}^1$ via the trivial action on E and the usual scaling action on $\mathbf{P}^1$. We give $\mathcal{V}$ a $\mathbb{C}^*$-equivariant structure as follows: For $-2 \leq i \leq 0$, give $\mathcal{O}(i)$ the natural $\mathbb{C}^*$ structure in which $\mathbb{C}^*$ acts with weight zero on the fibre at infinity. Then take the natural $\mathbb{C}^*$ action this induces on the components of $\mathcal{V}$.

We regard $\mathcal{V}$ as a family of equivariant C^∞ vector bundles on E parametrized by $\mathbf{P}^1$ and will write $\mathcal{V}_z$ for the restriction of $\mathcal{V}$ to $E \times \{z\}$. We will also want to think of $\mathcal{V}$ as a sheaf living on $\mathbf{P}^1$, in which case we write it as $B_{\mathcal{V}}$. Thus, for an open subset $U \subset \mathbf{P}^1$, the sections $B_{\mathcal{V}}(U)$ consist of the C^∞ sections of $\mathcal{V}|_{E\times U}$. Let $L_{\mathcal{V}}$ denote the sheaf of (infinite-dimensional) Lie algebras on $\mathbf{P}^1$ whose sections $L_{\mathcal{V}}(U)$ are defined to be the space of C^∞ sections of the vector bundle $\mathrm{End}(\mathcal{V})|_{E\times U}$.

As in [EF], the sheaf $L_{\mathcal{V}}$ has a sheaf of central extensions $\widehat{L}_{\mathcal{V}}$ that, as an extension of sheaves, has the form

$$0 \to C^\infty(\mathbf{P}^1, \mathbb{C}) \to \widehat{L}_{\mathcal{V}} \to L_{\mathcal{V}} \to 0.$$

In the topologically trivial case $\widehat{L}_{\mathcal{V}}$ is the central extension determined by the cocycle $\Omega(X,Y) = \int_E \eta \wedge \langle X, \overline{\partial}_t Y\rangle$, where η is a nonzero holomorphic 1-form on E and $\langle\, ,\, \rangle$ is the trace form. For general $\mathcal{V}$, one needs to replace the symbol $\overline{\partial}_t$ by any partial $\overline{\partial}$-operator in the E direction. For example, this may be done by choosing a $\overline{\partial}$-operator for $\mathcal{V}$ on $E \times \mathbf{P}^1$, say $\overline{D} : \mathcal{V} \to \mathcal{V} \otimes (T^*)^{0,1}(E \times \mathbf{P}^1)$, and using the splitting $T^*(E \times \mathbf{P}^1) \cong p_E^* T^* E \oplus p_{\mathbf{P}^1}^* T^* \mathbf{P}^1$ to project to an operator $\overline{\partial} : \mathcal{V} \to \mathcal{V} \otimes p_E^*(T^*)^{0,1}(E)$. A useful intuition is to think of $\widehat{L}_{\mathcal{V}}$ as the sheaf of sections of a C^∞-bundle of centrally extended current algebras.

Write $E = \mathbb{C}/\Lambda$ and let t denote the coordinate on $\mathbb{C}$ as well as the induced local coordinate on E. Consider a first-order differential operator D on $\mathcal{V}$ that locally, say on an open subset $E \times U \subset E \times \mathbf{P}^1$ with coordinates t on E and z on $\mathbf{P}^1$, has the form $D = \lambda \cdot d\bar{t} \cdot \partial/\partial\bar{t} + A \cdot d\bar{t}$, where A is a matrix of endomorphisms of $\mathcal{V}$ with coefficients that are C^∞ functions on $E \times U$ and $\lambda \in \mathbb{C}$. Let $\widehat{L}^*_{\mathcal{V}}$ denote the space of such operators, regarded as a sheaf over $\mathbf{P}^1$. This bundle is essentially (in a sense we will not try to make precise) the smooth part of the fibrewise dual of $\widehat{L}_{\mathcal{V}}$ (see [EF] for a discussion). We will therefore regard $\widehat{L}^*_{\mathcal{V}}$ with the action of the sheaf of groups of automorphisms of $\mathcal{V}$ as the bundle of coadjoint representations of the centrally extended relative gauge group of $\mathcal{V}$ whose associated centrally extended endomorphism algebra is $\widehat{L}_{\mathcal{V}}$. Finally, let $(\widehat{L}^*_{\mathcal{V}})_\lambda$ denote the bundle of hyperplanes in $\widehat{L}^*_{\mathcal{V}}$ consisting of those operators D having the given coefficient λ of $d\bar{t} \cdot \partial/\partial\bar{t}$.

Definition 9.12. A *$\overline{\partial}$-operator* $\overline{\partial}_B$ on $B_{\mathcal{V}}$ is an operator $\overline{\partial}_B : B_{\mathcal{V}} \to B_{\mathcal{V}} \otimes (T^*)^{0,1}\mathbf{P}^1$ on $\mathbf{P}^1$ that can be represented locally on $\mathbf{P}^1$ in the form $\overline{\partial}_B = d\bar{z} \cdot \partial/\partial\bar{z} + B \cdot d\bar{z}$ for some section B of $L_{\mathcal{V}}$.

A *C^∞ meromorphic Higgs field* Φ on $\mathbf{P}^1$ is a C^∞ section of $(T^*)^{1,0}(\mathbf{P}^1) \otimes \mathcal{O}(D) \otimes \widehat{L}^*_{\mathcal{V}}$, for some effective divisor D on $\mathbf{P}^1$. If one fixes a section s of $(T^*)^{0,1}(\mathbf{P}^1) \otimes \mathcal{O}(D)$, then it makes sense to talk of a meromorphic Higgs field lying in $s \otimes (\widehat{L}^*_{\mathcal{V}})_\lambda$.

In our case, we will fix the divisor $0 + \infty$ on $\mathbf{P}^1$, as well as coadjoint orbits in the fibres of $\widehat{L}_{\mathcal{V}}$ over 0 and ∞. By [EF, Proposition 3.2], these orbits correspond to isomorphism classes of holomorphic structures in $\mathcal{V}|_{E\times\{0\}}$ and $\mathcal{V}|_{E\times\{\infty\}}$, and we choose these to be the split bundle $(\mathcal{L}^*)^{d_{-1}} \oplus H \oplus \overline{\mathcal{L}}^{\,d_1}$ over E_0 and the trivial bundle of rank $d_{-1} + r + d_1$ over E_∞. As before, we make these bundles $\mathbb{C}^*$-equivariant by using weights $0, -1, -2$ for the factors of $\mathcal{V}$ along E_0 and weight 0 along E_∞.

The link between σ-monads on E and Higgs bundles on $\mathbf{P}^1$ is then the following correspondence.

Proposition 9.13. *Let $S = \mathrm{Skl}(E, \mathcal{L}, \sigma)$ and keep the above notation. Then there exists a bijective correspondence between isomorphism classes of*

1. *σ-monads $\mathbf{K}$ of the form (9.1) that satisfy $\mathrm{H}^0(\mathbf{K})|_E \cong H$, and*
2. *$\mathbb{C}^*$-equivariant meromorphic Higgs pairs $(\overline{\partial}_B, \Phi)$ on the C^∞-bundle $\mathcal{V}$ for which Φ lies in $z^{-1}dz \otimes (\widehat{L}^*_{\mathcal{V}})_1$ with residues at 0 and ∞ in the given coadjoint orbits.*

Proof. Since this result is tangential to the results of the paper we will only outline the proof.

Using a variant of the Rees construction, one can identify a σ-monad $\mathbf{K}$ with a holomorphic structure $D_{\mathcal{V}}$ on the equivariant vector bundle $\mathcal{V}$ over $E \times \mathbf{P}^1$. Now use the technique of [GM]: given a $\overline{\partial}$-operator $D_{\mathcal{V}}$ on $\mathcal{V}$ over $E \times \mathbf{P}^1$, split it into components D_E and $D_{\mathbf{P}^1}$ taking values in $\mathcal{V} \otimes (T^*)^{0,1} E$, respectively $\mathcal{V} \otimes (T^*)^{0,1} \mathbf{P}^1$. The correspondence of the proposition then takes $D_{\mathcal{V}}$ to the pair for which $\overline{\partial}_B = D_{\mathbf{P}^1}$ and $\Phi = D_E \cdot z^{-1} dz$. One then only needs to check that the gauge groups act in the same way on the two sides and that $D_{\mathcal{V}} \circ D_{\mathcal{V}} = 0$ is equivalent to the two equations $\overline{\partial}_B \circ \overline{\partial}_B = 0$ and $\overline{\partial}_B(\Phi) = 0$. □

Index of Notation

Standard definitions concerning Qgr-S can be found on pages 8—10 and these will not be listed in this index.

References

[Aj] K. Ajitabh, Residue complex for Sklyanin algebras of dimension three, *Adv. in Math.* **144** (1999), 137-160.

[ASh] M. Artin and W. F. Schelter, Graded algebras of global dimension 3, *Adv. in Math.* **66** (1987), 171-216.

[ASZ] M. Artin, L. W. Small and J. J. Zhang, Generic flatness for strongly noetherian algebras, *J. Algebra* **221** (1999), 579-610.

[AS] M. Artin and J. T. Stafford, Noncommutative graded domains with quadratic growth, *Invent. Math.,* **66** (1995), 231-276.

[ATV1] M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, in *The Grothendieck Festschrift*, vol. 1, Birkhäuser, Boston, 1990, pp. 33–85.

[ATV2] ———, Modules over regular algebras of dimension 3, *Invent. Math.* **106** (1991), 335–388.

[AV] M. Artin and M. Van den Bergh, Twisted homogeneous coordinate rings, *J. Algebra* **133** (1990), 249–271.

[AZ1] M. Artin and J. J. Zhang, Noncommutative projective schemes, *Adv. in Math.* **109** (1994), no. 2, 228–287.

[AZ2] ———, Abstract Hilbert schemes, *Algebr. Represent. Theory* **4** (2001), 305–394.

[BGK1] V. Baranovsky, V. Ginzburg and A. Kuznetsov, Quiver varieties and a non-commutative P^2, *Compositio Math.* **134** (2002), 283–318.

[BGK2] ———, Wilson's Grassmannian and a noncommutative quadric, *Int. Math. Res. Not.* **2003:21** (2003), 1155-1197.

[Be] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.* **18** (1983), 755–782.

[BJL] D. Berenstein, V. Jejjala and R. G. Leigh, Marginal and relevant deformations of $N = 4$ field theories and non-commutative moduli spaces of vacua, *Nuclear Phys. B* **589** (2000), 196–248.

[BL] D. Berenstein and R. G. Leigh, Resolution of stringy singularities by noncommutative algebras, *J. High Energy Phys.,* 2001, no. 6, Paper 30.

[BW1] Y. Berest and G. Wilson, Automorphisms and ideals of the Weyl algebra, *Math. Ann.* **318** (2000), 127–147.

[BW2] ———, Ideal classes of the Weyl algebra and noncommutative projective geometry. With an appendix by Michel Van den Bergh, *Int. Math. Res. Not.* (2002) **no. 26**, 1347–1396.

[CH] R. C. Cannings and M. P. Holland, Right ideals of rings of differential operators, *J. Algebra* **167** (1994), 116–141

[CDS] A. Connes, M. Douglas and A. Schwarz, Noncommutative geometry and matrix theory: compactification on tori. *J. High Energy Phys.* **1998**, no. 2, Paper 3, 35 pp.

[DV] K. de Naeghel and M. van den Bergh, Ideal classes of three dimensional Sklyanin algebras, Preprint, 2002.

[Di] J. Dixmier, Sur les alèbres de Weyl, *Bull. Soc. Math. France* **96** (1968), 209-242.

[DN] M. R. Douglas and N. A. Nekrasov, Noncommutative field theory, *Rev. Modern Phys.*, **73** (2001), 977–1029.

[DL] J. M. Drezet and J. Le Potier, Fibrés stables et fibrés exceptionnels sur $\mathbf{P}^2$, *Ann. Scient. Éc. Norm. Sup.*, **18** (1985), 193–244.

[Ei] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Math. 150, Springer-Verlag, New York, 1995.

[EF] P. I. Etingof and I. B. Frenkel, Central extensions of current groups in two dimensions, *Comm. Math. Phys.*, **165** (1994), 429–444.

[FO] B. L. Feigin and A. V. Odesskii, Vector bundles on an elliptic curve and Sklyanin algebras, In *Topics in quantum groups and finite-type invariants*, *Amer. Math. Soc. Transl.* **185** (1998), 65–84.

[GM] H. Garland and M. K. Murray, Kac-Moody monopoles and periodic instantons, *Comm. Math. Phys.* **120** (1988), no. 2, 335–351.

[Gi] V. Ginzburg, Non-commutative symplectic geometry, quiver varieties, and operads, *Math. Res. Lett.* **8** (2001), no. 3, 377–400.

[GN] A. Gorsky and N. Nekrasov, Elliptic Calogero-Moser system from two-dimensional current algebra. arXiv:hep-th/9401021.

[EGA] A. Grothendieck, Éléments de géométrie algébrique, Chapters III and IV, *Inst. Hautes Études Sci. Publ. Math.*, **11** (1961), **17** (1963), **20** (1964), **24** (1965), **28** (1966), **32** (1967).

[Ha] R. Hartshorne, *Algebraic geometry*, Springer Verlag, Berlin, 1977.

[HL] D. Huybrechts and M. Lehn, *The geometry of moduli spaces of sheaves*, Friedr. Vieweg & Son, Braunschweig, 1997.

[KKO] A. Kapustin, A. Kuznetsov and D. Orlov, Noncommutative instantons and twistor transform, *Comm. Math. Phys.* **221** (2001), 385–432.

[LB1] L. LeBruyn, Moduli spaces for right ideals of the Weyl algebra, *J. Algebra*, **172** (1995), 32–48.

[LB2] ———, *Noncommutative geometry at n*, in preparation.

[LP1] J. Le Potier, À propos de la construction de l'espace de modules des faisceaux semistables sur le plan projectif, *Bull. Soc. Math. France* **122** (1994), 363–369.

[LP2] ———, *Lectures on vector bundles*, CUP, Cambridge, 1997.

[Ma] Yu. I. Manin, *Quantum groups and noncommutative geometry*, Université de Montréal, Centre de Recherches Mathématiques, Montréal, 1988.

[MR] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, John Wiley & Sons, New York, 1987.

[Na] H. Nakajima, Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras, *Duke Math. J.* **76** (1994), 365–416.

[NV] C. Năstăsescu and F. Van Oystaeyen, *Graded ring theory*, North Holland, Amsterdam, 1982.

[Ne] N. Nekrasov, Infinite-dimensional algebras, many-body systems and gauge theories, in *Moscow seminars in mathematical physics*, *Amer. Math. Soc. Transl. Ser. 2* **191** (1999), 263–299.

[NSc] N. Nekrasov and A. Schwarz, Instantons on noncommutative $\mathbf{R}^4$ and $(2,0)$ superconformal six-dimensional theory, *Comm. Math. Phys.* **198** (1998), 689–703.

[Od] A. V. Odesskii, Elliptic algebras, *Russian Math. Surveys*, **75** (2002), 1127-1162.

[OSS] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, Progress in Math. 3, Birkhäuser, Boston, 1980.

[Pl] A. Polishchuk, Poisson structures and birational morphisms associated with bundles on elliptic curves, *Internat. Math. Res. Notices* **1998**, no. 13, 683–703.

[Pp] N. Popescu, *Abelian categories with applications to rings and modules*, Academic Press, London, 1973.

[RSS] R. Resco, L.W. Small and J. T. Stafford, Krull and global dimension of semiprime noetherian PI rings, *Trans. Amer. Math. Soc.* **274** (1982), 285-296.

[Rg] D. Rogalski, Examples of generic noncommutative surfaces, *Adv. in Math.*, to appear. arXiv:math.RA/0203180.

[Ru] A. Rudakov, Stability for an abelian category, *J. Algebra* **197** (1997), 231–245.

[SW] N. Seiberg and E. Witten, String theory and noncommutative geometry, *J. High Energy Phys.* **1999**, no. 9, Paper 32, 93 pp.

[Se] C. S. Seshadri, Geometric reductivity over arbitrary base, *Adv. in Math.* **26** (1977), 225–274.

[Si] C. Simpson, Moduli of representations of the fundamental group of a smooth projective variety I, *Publ. Math. IHES*, **79** (1994), 47–129.

[Sm] S. P. Smith, Some finite-dimensional algebras related to elliptic curves, in *Representation theory of algebras and related topics (Mexico City, 1994)*, 315–348, CMS Conf. Proc., 19, Amer. Math. Soc., Providence, RI, 1996.

[St1] J. T. Stafford, Stably free projective right ideals, *Compositio Math.*, **54** (1985), 63-78.

[St2] ———, Noncommutative projective geometry, *Proceedings of the ICM, Vol. II (Beijing, 2002)*, 93–103.

[SV] J. T. Stafford and M. Van den Bergh, Noncommutative curves and noncommutative surfaces, *Bull. Amer. Math. Soc.* **38** (2001), 171-216.

[TV] J. Tate and M. Van den Bergh, Homological properties of Sklyanin algebras, *Invent. Math.* **124** (1996), 619–647.

[VW1] F. Van Oystaeyen and L. Willaert, Cohomology of schematic algebras, *J. Algebra* **185** (1996), 74-84.

[VW2] ———, Examples and quantum sections of schematic algebras, *J. Pure Appl. Algebra* **120** (1997), 195–211.

[Ve] A. B. Verevkin, On a non-commutative analogue of the category of coherent sheaves on a projective scheme, *Amer. Math. Soc. Transl.* **151** (1992), 41–53.

[Wi] G. Wilson, Collisions of Calogero-Moser particles and an adelic Grassmannian, *Invent. Math.* **133** (1998), 1–41.

[YZ] A. Yekutieli and J. J. Zhang, Serre duality for noncommutative projective schemes, *Proc. Amer. Math. Soc.* **125** (1997), 697–707.

[Zh] J. J. Zhang, Connected graded Gorenstein rings with enough normal elements, *J. Algebra*, **189** (1997), 390–405

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